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On Begins, Meets and Before

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Interval temporal logics (ITLs) are logics for reasoning about temporal statements expressed over intervals, i.e., periods of time. The most famous temporal logic for intervals studied so far is probably Halpern and Shoham's HS, which is the logic of the thirteen Allen's interval relations. Unfortunately, HS and most of its fragments have an undecidable satisfiability problem. This discouraged the research in this area until recently, when a number non-trivial decidable ITLs have been discovered. This paper is a contribution towards the complete classification of all different fragments of HS. We consider different combinations of the interval relations *begins* (B), *meets* (A), *later* (L) and their inverses $\overline{\mathsf{A}}$, $\overline{\mathsf{B}}$ and $\overline{\mathsf{L}}$. We know from previous work that the combination $\mathsf{A}\mathsf{B}\overline{\mathsf{B}}\overline{\mathsf{A}}$ is decidable only when finite domains are considered (and undecidable elsewhere), and that $\mathsf{A}\mathsf{B}\overline{\mathsf{B}}$ is decidable over the natural numbers. We extend these results by showing that decidability of $\mathsf{A}\mathsf{B}\overline{\mathsf{B}}$ can be further extended to capture the language $\mathsf{A}\mathsf{B}\overline{\mathsf{B}}\overline{\mathsf{L}}$, which lies in between $\mathsf{A}\mathsf{B}\overline{\mathsf{B}}$ and $\mathsf{A}\mathsf{B}\overline{\mathsf{B}}\overline{\mathsf{A}}$, and that turns out to be maximal w.r.t decidability over strongly discrete linear orders (e.g. finite orders, the naturals, the integers). We also prove that the proposed decision procedure is optimal with respect to the EXPSPACE complexity class, and that the language is powerful enough to polynomially encode metric constraints on the length of the current interval.

Keywords: Interval Temporal Logic; Decidability; Complexity.

1. Introduction

Interval temporal logics (ITLs) are logics for reasoning about temporal statements expressed over intervals instead of points. The most famous temporal logic for interval studied so far is probably Halpern and Shoham's HS [8], which is the logic of (the thirteen) Allen's interval relations between intervals [1]. It features a modal operator for each relation, that is *meets* ($\langle \mathsf{A} \rangle$) (sometimes called *after*), *begins* ($\langle \mathsf{B} \rangle$), *ends* ($\langle \mathsf{E} \rangle$), *overlaps* ($\langle \mathsf{O} \rangle$), *during* ($\langle \mathsf{D} \rangle$), *later* ($\langle \mathsf{L} \rangle$), and their inverses (denoted by $\langle \overline{\mathsf{X}} \rangle$, where $\langle \mathsf{X} \rangle$ is a modal operator), although some of them are definable in terms of others. Since HS is undecidable when interpreted over almost any interesting classes of linearly ordered sets, it is natural to ask whether there exist decidable fragments of it, and how the properties of the underlying linearly ordered domain can influence their decidable/undecidable status. In the literature, the classes of linear orderings that have received more attention are i) the class of all linearly ordered sets, ii) the set of all discrete linearly ordered sets, iii) the class of all dense linearly ordered sets. In the second case one can also distinguish among *strong* discreteness (i.e., $\mathbb{N}, \mathbb{Z}$-like), and *weak* discreteness (which allows non-standard models such as $\mathbb{N} + \mathbb{Z}$). In recent years, a number of papers have been published in which new, sometimes unexpected, decidable and undecidable fragments are presented. Among them, we mention the fragment $\mathsf{A}\bar{\mathsf{A}}$, also known as PNL, presented in [6], and studied also in [4], which is decidable over all interesting classes of linear orders, and the fragment $\mathsf{AB}\bar{\mathsf{B}}$ (and, by symmetry, $\bar{\mathsf{A}}\mathsf{E}\bar{\mathsf{E}}$) which is decidable when interpreted over natural numbers [10]. Interestingly enough, the extension $\mathsf{AB}\bar{\mathsf{B}}\bar{\mathsf{A}}$ (and $\mathsf{AE}\bar{\mathsf{E}}\bar{\mathsf{A}}$) turns out to be decidable only when finite models are considered, and undecidable as soon as an infinite ascending (resp., descending) chain is admitted [9]. Other interesting fragments are $\mathsf{B}\bar{\mathsf{B}}$ and $\mathsf{E}\bar{\mathsf{E}}$, that are decidable in most cases [7], while any other combination of the four operators B, $\bar{\mathsf{B}}$, E, and $\bar{\mathsf{E}}$ immediately leads to undecidability [2]. Other combinations such as $\bar{\mathsf{A}}\mathsf{B}\bar{\mathsf{B}}$, and the simpler $\bar{\mathsf{A}}\mathsf{B}$, though, remain still uncovered.

In this paper, we present another piece of this complicated puzzle by considering also the Allen's relation *before*, that captures any interval ending at some point before the beginning point of the current interval, and it can be defined as $\langle \overline{\mathsf{A}} \rangle \langle \overline{\mathsf{A}} \rangle$. We will show that the logic $\mathsf{AB}\bar{\mathsf{B}}\bar{\mathsf{L}}$ (and the symmetric logic $\bar{\mathsf{A}}\mathsf{E}\bar{\mathsf{E}}\mathsf{L}$) is decidable (precisely, EXPSPACE-complete) when interpreted in the class of all strongly discrete linear orders. It is worth emphasizing that adding any other non-definable Allen's relation to $\mathsf{AB}\bar{\mathsf{B}}\bar{\mathsf{L}}$ and to $\bar{\mathsf{A}}\mathsf{E}\bar{\mathsf{E}}\mathsf{L}$ leads to undecidability over all considered structures (with the exception of $\bar{\mathsf{A}}$ and A, respectively, over finite orders). Hence, our results shows also that $\mathsf{AB}\bar{\mathsf{B}}\bar{\mathsf{L}}$ and $\bar{\mathsf{A}}\mathsf{E}\bar{\mathsf{E}}\mathsf{L}$ are maximal fragments of HS with respect to decidability in the class of all strongly discrete linear orders. We focus our attention on the so-called *strict* semantics, thus excluding point-intervals from our models, but all results can be easily extended to include them. Moreover, we discuss the relationship between the logic $\mathsf{AB}\bar{\mathsf{B}}\bar{\mathsf{L}}$ and some metric extension of interval temporal logics recently presented in the literature [5, 3]: we show that we are able to embed metric constraints on the length of intervals by means of a non-trivial, polynomial encoding, which is somehow surprising given that the intuitive embedding is exponential.

The structure of this paper is as follows. In Section 2 we introduce syntax and semantics of our logic. In Section 3, we discuss the decidability of the satisfiability problem over finite and infinite structures, while in Section 4 we discuss its complexity. In Section 5 we show how to polynomially encode metric constraints on the length of intervals in $\mathsf{AB}\bar{\mathsf{B}}\bar{\mathsf{L}}$. Finally, in Section 6 we draw some conclusions and outline future research directions.

2. The interval temporal logic $\mathsf{AB\bar{B}\bar{L}}$

In this section, we briefly introduce syntax and semantics of the logic $\mathsf{AB\bar{B}\bar{L}}$, and we compare its expressiveness with two related interval temporal logics recently introduced in the literature. Then, we give the basic notions of atom, type, and dependency. We conclude the section by providing an alternative interpretation of $\mathsf{AB\bar{B}\bar{L}}$ over labeled grid-like structures.

2.1. *Syntax and semantics*

The logic $\mathsf{AB\bar{B}\bar{L}}$ features four modal operators $\langle A\rangle$, $\langle B\rangle$, $\langle \bar{B}\rangle$ and $\langle \bar{L}\rangle$, and it is interpreted over interval temporal structures based on strongly discrete linear orders endowed with the four Allen's relations A ("meets"), B ("begins"), $\bar{B}$ ("begun by") and $\bar{L}$ ("before"). We recall that a linear order $\mathbb{O} = \langle O, <\rangle$ is *strongly discrete* if and only if there are only finitely many points between any pair of points $x < y \in O$. Example of strongly discrete linear orders are all finite linear orders, and the sets $\mathbb{N}$ and $\mathbb{Z}$.

Given a set $\mathcal{P}rop$ of propositional variables, formulas of $\mathsf{AB\bar{B}\bar{L}}$ are built up from $\mathcal{P}rop$ using the boolean connectives $\neg$ and $\vee$ and the unary modal operators $\langle A\rangle$, $\langle B\rangle$, $\langle \bar{B}\rangle$, $\langle \bar{L}\rangle$. As usual, we shall take advantage of shorthands like $\varphi_1 \wedge \varphi_2 = \neg(\neg\varphi_1 \vee \neg\varphi_2)$, $[A]\varphi = \neg\langle A\rangle\neg\varphi$, $[B]\varphi = \neg\langle B\rangle\neg\varphi$, etc. Hereafter, we denote the size of φ by $|\varphi|$. Given any strongly discrete linear order $\mathbb{O} = \langle O, <\rangle$ we define $\mathbb{I}_{\mathbb{O}}$ as the set of all closed intervals $[x, y]$, with $x, y \in O$ and $x < y$. For any pair of intervals $[x, y], [x', y'] \in \mathbb{I}_{\mathbb{O}}$, the Allen's relations "meets" A, "begins" B, "begun by" $\bar{B}$, and "before" $\bar{L}$ are defined as follows:

- **"meets" relation:** $[x, y]\ A\ [x', y']$ iff $y = x'$;
- **"begins" relation:** $[x, y]\ B\ [x', y']$ iff $x = x'$ and $y' < y$;
- **"begun by" relation:** $[x, y]\ \bar{B}\ [x', y']$ iff $x = x'$ and $y < y'$;
- **"before" relation:** $[x, y]\ \bar{L}\ [x', y']$ iff $y' < x$.

Given an *interval structure* $\mathcal{S} = (\mathbb{I}_{\mathbb{O}}, A, B, \bar{B}, \bar{L}, \sigma)$, where $\sigma : \mathbb{I}_{\mathbb{O}} \to \mathscr{P}(\mathcal{P}rop)$ is a labeling function that maps intervals in $\mathbb{I}_{\mathbb{O}}$ to sets of propositional variables, and an initial interval $I = [x, y]$, we define the semantics of an $\mathsf{AB\bar{B}\bar{L}}$ formula as follows:

- $\mathcal{S}, I \models a$ iff $a \in \sigma(I)$, for any $a \in \mathcal{P}rop$;
- $\mathcal{S}, I \models \neg\varphi$ iff $\mathcal{S}, I \not\models \varphi$;
- $\mathcal{S}, I \models \varphi_1 \vee \varphi_2$ iff $\mathcal{S}, I \models \varphi_1$ or $\mathcal{S}, I \models \varphi_2$;
- for every relation $R \in \{A, B, \bar{B}, \bar{L}\}$, $\mathcal{S}, I \models \langle R\rangle\varphi$ iff there is an interval $J \in \mathbb{I}_{\mathbb{O}}$ such that $I\ R\ J$ and $\mathcal{S}, J \models \varphi$.

Given an interval structure $\mathcal{S}$ and a formula φ, we say that $\mathcal{S}$ *satisfies* φ (and hence φ is *satisfiable*) if there is an interval I in $\mathcal{S}$ such that $\mathcal{S}, I \models \varphi$. Accordingly, we define the *satisfiability problem* for $\mathsf{AB\bar{B}\bar{L}}$ as the problem of establishing whether a given $\mathsf{AB\bar{B}\bar{L}}$-formula φ is satisfiable.

2.2. *Expressivity*

Focusing our attention on strongly discrete linear orderings only, we compare $\mathsf{AB\bar{B}\bar{L}}$ with two other fragments of Halpern and Shoam's HS whose decidability has been recently

$\mathsf{AB\bar{B}\bar{A}}$ (Decidable only over finite structures [9])

$\mathsf{AB\bar{B}\bar{L}}$ (Decidable over finite and strongly discrete structures [this paper])

$\mathsf{AB\bar{B}}$ (Decidable over finite and strongly discrete structures [10])

Figure 1. Expressive power of some fragments of HS.

studied in the literature. The first result we consider has been stated in [10], where the fragment $\mathsf{AB\bar{B}}$ has been studied and shown to be decidable over strongly discrete linear oderings. Later on, the extension that includes also $\langle\bar{\mathsf{A}}\rangle$ has been analyzed in [9], where it has been proved that decidability still hold, but only over finite structures, therefore leaving as open the problem of establishing if there exists a fragment that locates itself between those two, and it is still decidable over infinite structures.

This is exactly the case of the fragment $\mathsf{AB\bar{B}\bar{L}}$: on the one side, it is easy to see that it can be embedded into $\mathsf{AB\bar{B}\bar{A}}$, since $\mathcal{S}, \mathrm{I} \models \langle\bar{\mathsf{L}}\rangle\varphi$ iff $\mathcal{S}, \mathrm{I} \models \langle\bar{\mathsf{A}}\rangle\langle\bar{\mathsf{A}}\rangle\varphi$ for every structure $\mathcal{S}$, and on the other side, since the modal operators $\langle\mathsf{A}\rangle$, $\langle\mathsf{B}\rangle$ and $\langle\bar{\mathsf{B}}\rangle$ allow the language to capture only intervals whose endpoints are greater or equals to the endpoints of the interval were a formula is interpreted, it is strictly more expressive than $\mathsf{AB\bar{B}}$. As a consequence of the results presented in this paper, we can conclude that $\mathsf{AB\bar{B}\bar{L}}$ is strictly less expressive than $\mathsf{AB\bar{B}\bar{A}}$, since the former is decidable in a wider class of linear orderings than the latter.

2.3. *Atoms, types, and dependencies*

Let $\mathcal{S} = (\mathbb{I}_{\mathbb{O}}, \mathsf{A}, \mathsf{B}, \bar{\mathsf{B}}, \bar{\mathsf{L}}, \sigma)$ be an interval structure that satisfies the $\mathsf{AB\bar{B}\bar{L}}$-formula φ. In the sequel, we relate intervals in $\mathcal{S}$ with respect to the set of sub-formulas of φ they satisfy. To do that, we introduce the key notions of φ-*atom* and φ-*type*.

First of all, we define the *closure* $\mathcal{C}l(\varphi)$ of φ as the set of all sub-formulas of φ and of their negations (we identify $\neg\neg\alpha$ with α, $\neg\langle\mathsf{A}\rangle\alpha$ with $[\mathsf{A}]\neg\alpha$, etc.). For technical reasons, we also introduce the *extended closure* $\mathcal{C}l^+(\varphi)$, which is defined as the union of $\mathcal{C}l(\varphi)$ with the set of all formulas of the forms $\langle\mathsf{R}\rangle\alpha$ and $\neg\langle\mathsf{R}\rangle\alpha$, with $\mathsf{R} \in \{\mathsf{A}, \mathsf{B}, \bar{\mathsf{B}}, \bar{\mathsf{L}}\}$ and $\alpha \in \mathcal{C}l(\varphi)$. A φ-*atom* is any non-empty set $\mathrm{F} \subseteq \mathcal{C}l^+(\varphi)$ such that (i) for every $\alpha \in \mathcal{C}l^+(\varphi)$, we have $\alpha \in \mathrm{F}$ iff $\neg\alpha \notin \mathrm{F}$ and (ii) for every $\gamma = \alpha \vee \beta \in \mathcal{C}l^+(\varphi)$, we have $\gamma \in \mathrm{F}$ iff $\alpha \in \mathrm{F}$ or $\beta \in \mathrm{F}$ (intuitively, a φ-*atom* is a maximal *locally consistent* set of formulas chosen from $\mathcal{C}l^+(\varphi)$). Note that the cardinalities of both sets $\mathcal{C}l(\varphi)$ and $\mathcal{C}l^+(\varphi)$ are *linear* in the number $|\varphi|$ of sub-formulas of φ, while the number of φ-atoms is *at most exponential* in $|\varphi|$ (precisely, we have $|\mathcal{C}l(\varphi)| = 2|\varphi|$, $|\mathcal{C}l^+(\varphi)| = 18|\varphi|$, and there are at most $2^{9|\varphi|}$ distinct atoms). We define $\mathcal{A}_\varphi$ as the set of all possible atoms that can be built over $\mathcal{C}l^+(\varphi)$.

We associate the set of all formulas $\alpha \in \mathcal{C}l^+(\varphi)$ such that $\mathcal{S}, \mathrm{I} \models \alpha$ with each interval $\mathrm{I} \in \mathcal{S}$. Such a set is called φ-*type* of I and it is denoted by $\mathcal{T}ype_{\mathcal{S}}(\mathrm{I})$. We have that every φ-type is a φ-atom, but not vice versa. Hereafter φ-atoms (resp., φ-types) will be simply called atoms (resp., types). Given an atom F, we denote by $\mathcal{O}bs(\mathrm{F})$ the set of all *observable* of F, namely, the formulas $\alpha \in \mathcal{C}l(\varphi)$ such that $\alpha \in \mathrm{F}$. Similarly, given an atom F and a

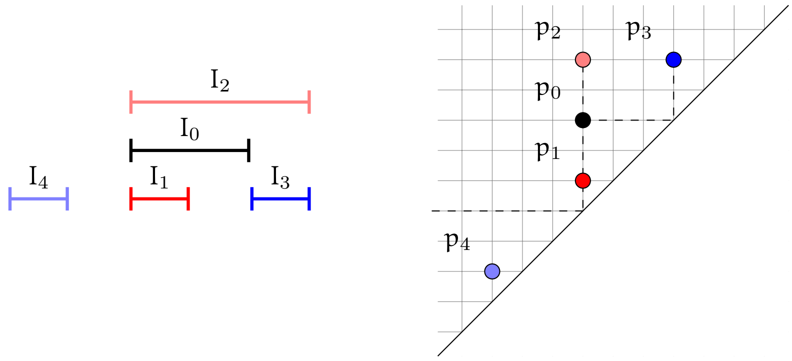

Figure 2. Correspondence between intervals and the points of a grid.

relation $R \in \{A, B, \bar{B}, \bar{L}\}$, we denote by $\mathcal{R}eq_R(F)$ the set of all R-*requests* of F, namely, the formulas $\alpha \in \mathcal{C}l(\varphi)$ such that $\langle R \rangle \alpha \in F$. Taking advantage of the above sets, we can define the following three relations between two atoms F and G:

$$F \xrightarrow{A} G \ \text{ iff } \mathcal{R}eq_A(F) \ = \ \mathcal{O}bs(G) \cup \mathcal{R}eq_B(G) \cup \mathcal{R}eq_{\bar{B}}(G)$$

$$F \xrightarrow{B} G \ \text{ iff } \begin{cases} \mathcal{O}bs(F) \cup \mathcal{R}eq_{\bar{B}}(F) \ \subseteq \ \mathcal{R}eq_{\bar{B}}(G) \ \subseteq \ \mathcal{O}bs(F) \cup \mathcal{R}eq_{\bar{B}}(F) \cup \mathcal{R}eq_B(F) \\ \mathcal{O}bs(G) \cup \mathcal{R}eq_B(G) \ \subseteq \ \mathcal{R}eq_B(F) \ \subseteq \ \mathcal{O}bs(G) \cup \mathcal{R}eq_B(G) \cup \mathcal{R}eq_{\bar{B}}(G) \\ \mathcal{R}eq_{\bar{L}}(F) = \mathcal{R}eq_{\bar{L}}(G). \end{cases}$$

$$F \xrightarrow{\bar{L}} G \ \text{ iff } \mathcal{O}bs(G) \cup \mathcal{R}eq_{\bar{L}}(G) \ \subseteq \ \mathcal{R}eq_{\bar{L}}(F)$$

Note that the relations $\xrightarrow{B}$ and $\xrightarrow{\bar{L}}$ are transitive, while $\xrightarrow{A}$ is not. Moreover, all $\xrightarrow{A}$, $\xrightarrow{B}$ and $\xrightarrow{\bar{L}}$ satisfy a *view-to-type dependency*, namely, for every pair of intervals I, J in $\mathcal{S}$, we have that

$$\begin{array}{lll} I \ A \ J & \text{implies} & \mathcal{T}ype_{\mathcal{S}}(I) \xrightarrow{A} \mathcal{T}ype_{\mathcal{S}}(J) \\ I \ B \ J & \text{implies} & \mathcal{T}ype_{\mathcal{S}}(I) \xrightarrow{B} \mathcal{T}ype_{\mathcal{S}}(J) \\ I \ \bar{L} \ J & \text{implies} & \mathcal{T}ype_{\mathcal{S}}(I) \xrightarrow{\bar{L}} \mathcal{T}ype_{\mathcal{S}}(J). \end{array}$$

2.4. *Compass structures*

The logic $AB\bar{B}\bar{L}$ can be equivalently interpreted over grid-like structures (hereafter called *compass structures* [12]) by exploiting the existence of a natural bijection between the intervals $I = [x, y]$ and the points $p = (x, y)$ of an $O \times O$ grid such that $x < y$. As an example, in Fig. 2 are shown five intervals $I_0, ..., I_4$, such that $I_0 \ B \ I_1$, $I_0 \ \bar{B} \ I_2$, $I_0 \ A \ I_3$, and $I_0 \ \bar{L} \ I_4$, together with the corresponding points $p_0, ..., p_4$ of a grid (note that the four Allen's relations $A, B, \bar{B}, \bar{L}$ between intervals are mapped to the corresponding spatial relations between points).

Definition 1. *Given an* $AB\bar{B}\bar{L}$ *formula* φ*, a (consistent and fulfilling)* compass φ-structure *is a pair* $\mathcal{G} = (\mathbb{P}_{\mathbb{O}}, \mathcal{L})$*, where* $\mathbb{P}_{\mathbb{O}}$ *is the set of points of the form* $p = (x, y)$*, with* $x, y \in O$

and $x < y$*, and* $\mathcal{L}$ *is function that maps any point* $p \in \mathbb{P}_{\mathbb{O}}$ *to a* φ*-atom* $\mathcal{L}(p)$ *in such a way that:*

- *for every pair of points* $p, q \in \mathbb{P}_{\mathbb{O}}$ *and every relation* $R \in \{A, B, \bar{L}\}$*, if* $p \; R \; q$ *holds, then* $\mathcal{L}(p) \xrightarrow{R} \mathcal{L}(q)$ *follows (**consistency**);*
- *for every point* $p \in \mathbb{P}_{\mathbb{O}}$*, every relation* $R \in \{A, B, \bar{B}, \bar{L}\}$*, and every formula* $\alpha \in \mathcal{R}eq_R(\mathcal{L}(p))$*, there is a point* $q \in \mathbb{P}_{\mathbb{O}}$ *such that* $p \; R \; q$ *and* $\alpha \in \mathcal{O}bs(\mathcal{L}(q))$ *(**fulfillment**).*

We say that a compass φ-structure $\mathcal{G} = (\mathbb{P}_{\mathbb{O}}, \mathcal{L})$ *features* a formula α if there is a point $p \in \mathbb{P}_{\mathbb{O}}$ such that $\alpha \in \mathcal{L}(p)$. The following proposition implies that the satisfiability problem for $AB\bar{B}\bar{L}$ is reducible to the problem of deciding, for any given formula φ, whether there exists a φ-compass structure featuring φ.

Proposition 2. *An* $AB\bar{B}\bar{L}$*-formula* φ *is satisfied by some interval structure if and only if it is featured by some* φ*-compass structure.*

3. Deciding the satisfiability problem for $AB\bar{B}\bar{L}$

In this section, we prove that the satisfiability problem for $AB\bar{B}\bar{L}$ is decidable by providing a small-model theorem for it. For the sake of simplicity, we first show that the satisfiability problem for $AB\bar{B}\bar{L}$ interpreted over *finite* interval structures is decidable and then we extend such a result to infinite interval structures based on strong discrete linear orders.

As a preliminary step, we introduce the key notions of shading, of witness set, and of compatibility between rows of a compass structure. Let $\mathcal{G} = (\mathbb{P}_{\mathbb{O}}, \mathcal{L})$ be a compass structure and let $y \in O$. The *shading of the row* y, denoted by $\mathcal{S}ha_{\mathcal{G}}(y)$, is the defined as the set $\mathcal{S}ha_{\mathcal{G}}(y) = \{\mathcal{L}(x, y) : x < y\}$, namely, the set of the atoms of all points in $\mathbb{P}_{\mathbb{O}}$ whose vertical coordinate has value y. Similarly, a *witness set for* y, denoted by $\mathcal{W}it(y)$, is any minimal set $\mathcal{W}it(y) \subseteq \{(x_\psi, y_\psi) : x_\psi < y_\psi \wedge y_\psi > y\}$ that respects the following property: for every $\psi \in \mathcal{C}l(\varphi)$ that appears in the labeling of some point (x', y') with $y' > y$, there exists a *witness* $(x_\psi, y_\psi) \in \mathcal{W}it(y)$ such that:

(1) $\psi \in \mathcal{L}(x_\psi, y_\psi)$, and

(2) y_ψ is minimal, that is, for all (x', y') with $y < y' < y_\psi$, $\neg\psi \in \mathcal{L}(x', y')$.

Since $\mathcal{W}it(y)$ is minimal we have that there is at most one point for every $\psi \in \mathcal{C}l(\varphi)$ and thus $|\mathcal{W}it(y)| < |\mathcal{C}l(\varphi)| = 2 \cdot |\varphi|$. Intuitively, a witness set for a row y is a set that contains a witness (x_ψ, y_ψ) for every formula ψ that occurs in some point above the row y, that is, a point that satisfies ψ such that the distance $y_\psi - y$ is minimal.

Let $P \subseteq \mathbb{P}_{\mathbb{O}}$ a set of points and let $y \in O$ be a row of the compass structure. We define the *projection of* P *on the row* y (and we denote it by $\pi_y(P)$) as the set $\pi_y(P) = \{x : (x, y) \in P \wedge x < y)\}$. The projection operator, paired with the notion of shading and of witness set, allows us to determine whether two rows are *compatible* or not.

Definition 3. *Given a compass structure* $\mathcal{G}$ *and two rows* $y_0 < y_1$*, we say that* y_0 *and* y_1 *are* compatible *if and only if the following properties holds:*

(1) $\mathcal{S}ha_{\mathcal{G}}(y_0) = \mathcal{S}ha_{\mathcal{G}}(y_1)$*;*

(2) $\mathcal{L}(y_0 - 1, y_0) = \mathcal{L}(y_1 - 1, y_1)$*;*

(3) there exists a witness set $\mathcal{W}it(y_1)$ *for* y_1 *and an* injective mapping function $w : \pi_{y_1}(\mathcal{W}it(y_1)) \mapsto \{x : x < y_0\}$ *s.t.* $\mathcal{L}(x, y_1) = \mathcal{L}(w(x), y_0)$ *for every* $x \in \pi_{y_1}(\mathcal{W}it(y_1))$*, that assigns a point on the row* y_0 *for every witness* (x_ψ, y_ψ) *in* $\mathcal{W}it(y_1)$ *with* $x_\psi \leqslant y_1$.

Notice that, given two witnesses $(x_\psi, y_\psi), (x_\xi, y_\xi) \in \mathcal{W}it(y_1)$ such that $x_\psi = x_\xi$, by the definition of the projection operator, the function w assigns the same point $(w(x), y_0)$ to both of them.

3.1. *A small-model theorem for finite structures*

Let φ be an $\mathsf{AB\bar{B}\bar{L}}$ formula. It is easy to see that φ is satisfiable over a finite model if and only if the formula $\varphi \vee \langle \bar{B} \rangle \varphi \vee \langle A \rangle \varphi \vee \langle A \rangle \langle A \rangle \varphi$ is featured by the *initial point* $(0, 1)$ of a *finite* compass structure $\mathcal{G} = (\mathbb{P}_{\mathbb{O}}, \mathcal{L})$.

In the following, given a compass structure $\mathcal{G} = (\mathbb{P}_{\mathbb{O}}, \mathcal{L})$, we refer to the number of points in $\mathbb{O}$ as the *size of* $\mathcal{G}$. We will prove that we can restrict our attention to compass structures whose size is bounded by a double exponential in $|\varphi|$. We start with the following lemma that proves two simple, but crucial, properties of the relations $\xrightarrow{A}$, $\xrightarrow{B}$, and $\xrightarrow{\bar{L}}$.

Lemma 4. *Let* F, G, H *be some atoms:*

(1) if $F \xrightarrow{A} H$ *and* $G \xrightarrow{B} H$ *hold, then* $F \xrightarrow{A} G$ *holds as well;*

(2) if $F \xrightarrow{B} G$ *and* $G \xrightarrow{\bar{L}} H$ *hold, then* $F \xrightarrow{\bar{L}} H$ *holds as well.*

Proof. The proof for property 1 can be found in [10]. As for property 2, we have that, by the definition of $\xrightarrow{B}$, if $F \xrightarrow{B} G$ then $\mathcal{R}eq_{\bar{L}}(F) = \mathcal{R}eq_{\bar{L}}(G)$. This implies that $\mathcal{O}bs(H) \cup \mathcal{R}eq_{\bar{L}}(H) \subseteq \mathcal{R}eq_{\bar{L}}(F)$ and thus $F \xrightarrow{\bar{L}} H$ holds as well. □

The next lemma shows that, under suitable conditions, a given compass structure $\mathcal{G}$ may be reduced in length, preserving the existence of atoms featuring φ.

Lemma 5. *Let* $\mathcal{G}$ *be a finite compass structure of size* N *featuring* φ *on the initial point* $(0, 1)$*. If there exist two compatible rows* $0 < y_0 < y_1 < N$ *in* $\mathcal{G}$*, then there exists a compass structure* $\mathcal{G}'$ *of size* $N' = N - y_1 + y_0$ *that features* φ.

Proof. Suppose that $0 < y_0 < y_1 < N$ are two compatible rows of $\mathcal{G}$. By Definition 3, we have that $\mathcal{S}ha_{\mathcal{G}}(y_0) = \mathcal{S}ha_{\mathcal{G}}(y_1)$, $\mathcal{L}(y_0 - 1, y_0) = \mathcal{L}(y_1 - 1, y_1)$, and there exists a witness set $\mathcal{W}it(y_1)$ for y_1 and an injective mapping function $w : \pi_{y_1}(\mathcal{W}it(y_1)) \mapsto \{x : x < y_0\}$. Then, we can define a function $f : \{0, ..., y_0 - 1\} \mapsto \{0, ..., y_1 - 1\}$ such that, for every $0 \leqslant x < y_0$, $\mathcal{L}(x, y_0) = \mathcal{L}(f(x), y_1)$ and for every $(x_\psi, y_\psi) \in \mathcal{W}it(y_1)$ if $x_\psi < y_1$ then $f(w(x_\psi)) = x_\psi$.

Let $k = y_1 - y_0$, $N' = N - k$, $\mathbb{O}' = \langle \{0, \ldots, N' - 1\}, < \rangle$, and $\mathbb{P}_{\mathbb{O}'}$ be the correspondent portion of the grid. We extend f to a function that maps points in $\mathbb{P}_{\mathbb{O}'}$ to points in $\mathbb{P}_{\mathbb{O}}$ as follows:

- if $p = (x, y)$, with $0 \leqslant x < y < y_0$, then we simply let $f(p) = p$;
- if $p = (x, y)$, with $0 \leqslant x < y_0 \leqslant y$, then we let $f(p) = (f(x), y + k)$;
- if $p = (x, y)$, with $y_0 \leqslant x < y$, then we let $f(p) = (x + k, y + k)$.

We denote by $\mathcal{L}'$ the labeling of $\mathbb{P}_{\mathbb{O}'}$ such that, for every point $p \in \mathbb{P}_{\mathbb{O}'}$, $\mathcal{L}'(p) = \mathcal{L}(f(p))$ and we denote by $\mathcal{G}'$ the resulting structure $(\mathbb{P}_{\mathbb{O}'}, \mathcal{L}')$ (see Figure 3). We have to prove that $\mathcal{G}'$ is a consistent and fulfilling compass structure that features φ. First, we show that $\mathcal{G}'$ satisfies the consistency conditions for the relations B, A, and $\bar{L}$; then we show that $\mathcal{G}'$ satisfies the fulfillment conditions for the $\bar{B}$-, B-, A, and $\bar{L}$-requests; finally, we show that $\mathcal{G}'$ features φ.

CONSISTENCY WITH RELATION B. Consider two points $p = (x, y)$ and $p' = (x', y')$ in $\mathcal{G}'$ such that $p \ B \ p'$, i.e., $0 \leqslant x = x' < y' < y < N'$. We prove that $\mathcal{L}'(p) \xrightarrow{B} \mathcal{L}'(p')$ by distinguishing among the following three cases (notice that exactly one of such cases holds):

(1) $y < y_0$ and $y' < y_0$,

(2) $y \geqslant y_0$ and $y' \geqslant y_0$,

(3) $y \geqslant y_0$ and $y' < y_0$.

If $y < y_0$ and $y' < y_0$, then, by construction, we have $f(p) = p$ and $f(p') = p'$. Since $\mathcal{G}$ is a (consistent) compass structure, we immediately obtain $\mathcal{L}'(p) = \mathcal{L}(p) \xrightarrow{B} \mathcal{L}(p') = \mathcal{L}'(p')$.

If $y \geqslant y_0$ and $y \geqslant y_0$, then, by construction, we have either $f(p) = (f(x), y + k)$ or $f(p) = (x + k, y + k)$, depending on whether $x < y_0$ or $x \geqslant y_0$. Similarly, we have either $f(p') = (f(x'), y' + k) = (f(x), y' + k)$ or $f(p') = (x' + k, y' + k) = (x + k, y' + k)$. This implies $f(p) \ B \ f(p')$ and thus, since $\mathcal{G}$ is a (consistent) compass structure, we have $\mathcal{L}'(p) = \mathcal{L}(f(p)) \xrightarrow{B} \mathcal{L}(f(p')) = \mathcal{L}'(p')$.

If $y \geqslant y_0$ and $y' < y_0$, then, since $x < y' < y_0$, we have by construction $f(p) = (f(x), y + k)$ and $f(p') = p'$. Moreover, if we consider the point $p'' = (x, y_0)$ in $\mathcal{G}'$, we easily see that (i) $f(p'') = (f(x), y_1)$, (ii) $f(p) \ B \ f(p'')$ (whence $\mathcal{L}(f(p)) \xrightarrow{B} \mathcal{L}(f(p''))$), (iii) $\mathcal{L}(f(p'')) = \mathcal{L}(p'')$, and (iv) $p'' \ B \ p'$ (whence $\mathcal{L}(p'') \xrightarrow{B} \mathcal{L}(p')$). It thus follows that $\mathcal{L}'(p) = \mathcal{L}(f(p)) \xrightarrow{B} \mathcal{L}(f(p'')) = \mathcal{L}(p'') \xrightarrow{B} \mathcal{L}(p') = \mathcal{L}(f(p')) = \mathcal{L}'(p')$. Finally, by exploiting the transitivity of the relation $\xrightarrow{B}$, we obtain $\mathcal{L}'(p) \xrightarrow{B} \mathcal{L}'(p')$.

CONSISTENCY WITH RELATION A. Consider two points $p = (x, y)$ and $p' = (x', y')$ such that $p \ A \ p'$, i.e., $0 \leqslant x < y = x' < y' < N'$. We define $p'' = (y, y + 1)$ in such a way that $p \ A \ p''$ and $p' \ B \ p''$ and we distinguish between the following two cases:

(1) $y \geqslant y_0$,

(2) $y < y_0$.

If $y \geqslant y_0$, then, by construction, we have $f(p) \ A \ f(p'')$. Since $\mathcal{G}$ is a (consistent) compass structure, it follows that $\mathcal{L}'(p) = \mathcal{L}(f(p)) \xrightarrow{A} \mathcal{L}(f(p'')) = \mathcal{L}'(p'')$.

If $y < y_0$, then, by construction, we have $\mathcal{L}(p'') = \mathcal{L}(f(p''))$. Again, since $\mathcal{G}$ is a (consistent) compass structure, it follows that $\mathcal{L}'(p) = \mathcal{L}(f(p)) = \mathcal{L}(p) \xrightarrow{A} \mathcal{L}(p'') = \mathcal{L}(f(p'')) = \mathcal{L}'(p'')$.

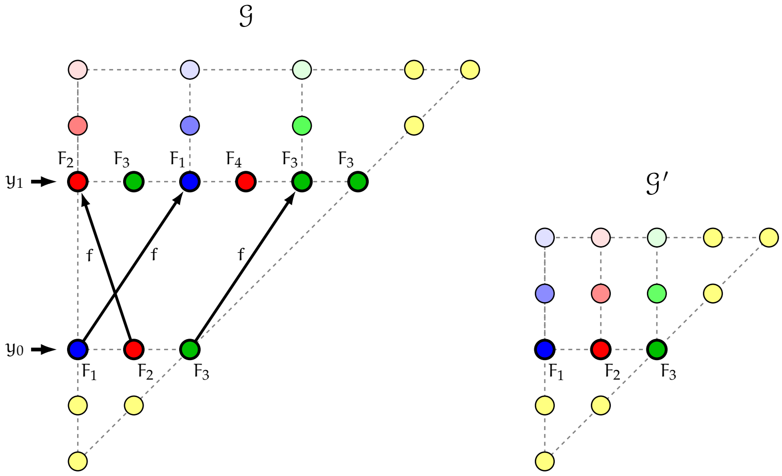

Figure 3. Contraction $\mathcal{G}'$ of a compass structure $\mathcal{G}$.

In both cases we have $\mathcal{L}'(p) \xrightarrow{A} \mathcal{L}'(p'')$. Now, we recall that p' B p'' and that, by previous arguments, $\mathcal{G}'$ is consistent with the relation B. We thus have $\mathcal{L}'(p') \xrightarrow{B} \mathcal{L}'(p'')$. Finally, by applying Lemma 4, we obtain $\mathcal{L}'(p) \xrightarrow{A} \mathcal{L}'(p')$.

CONSISTENCY WITH RELATION $\bar{L}$. Consider two points $p = (x, y)$ and $p' = (x', y')$ in $\mathcal{G}'$ such that $p\ \bar{L}\ p'$, i.e., $0 \leqslant x' < y' < x < y < N'$. We prove that $\mathcal{L}'(p) \xrightarrow{\bar{L}} \mathcal{L}'(p')$ by distinguishing among the following cases:

(1) $y < y_0$ and $y' < y_0$,

(2) $y \geqslant y_0$ and $y' \geqslant y_0$,

(3) $y \geqslant y_0$ and $y' < y_0$.

If $y < y_0$ and $y' < y_0$, then, by construction, we have $f(p) = p$ and $f(p') = p'$. Since $\mathcal{G}$ is consistent, we immediately obtain $\mathcal{L}'(p) = \mathcal{L}(p) \xrightarrow{\bar{L}} \mathcal{L}(p') = \mathcal{L}'(p')$.

If $y \geqslant y_0$ and $y' \geqslant y_0$, then, by construction, we have either $f(p') = (f(x'), y'+k)$ or $f(p') = (x'+k, y'+k)$, depending on whether $x' < y_0$ or $x' \geqslant y_0$. Since $y_0 \leqslant y' < x$, we have $f(p) = (x+k, y+k)$. This implies $f(p)\ \bar{L}\ f(p')$ and thus, since $\mathcal{G}$ is a (consistent) compass structure, we have $\mathcal{L}'(p) = \mathcal{L}(f(p)) \xrightarrow{\bar{L}} \mathcal{L}(f(p')) = \mathcal{L}'(p')$.

If $y \geqslant y_0$ and $y' < y_0$, then, we have by construction that $f(p') = p'$ and either $f(p) = (x+k, y+k)$ or $f(p) = (f(x), y+k)$. In the former case we have that $f(p)\ \bar{L}\ f(p')$ and thus, since $\mathcal{G}$ is a consistent compass structure, $\mathcal{L}'(p) = \mathcal{L}(f(p)) \xrightarrow{\bar{L}} \mathcal{L}(f(p')) = \mathcal{L}'(p')$. In the latter case it is not necessarily true that $y' < f(x)$. Consider the points $p'' = (f(x), y_1)$ and $p''' = (x, y_0)$: by the definition of f, $\mathcal{L}(p'') = \mathcal{L}(p''')$. Moreover, we have that $f(p) B p''$ and $p''' \bar{L} f(p') = p'$. Since $\mathcal{G}$ is a consistent compass structure, this implies that $\mathcal{L}'(p) = \mathcal{L}(f(p)) \xrightarrow{B} \mathcal{L}(p'') = \mathcal{L}(p''') \xrightarrow{\bar{L}} \mathcal{L}(f(p')) = \mathcal{L}'(p')$. Finally, by applying Lemma 4, we obtain $\mathcal{L}'(p) \xrightarrow{\bar{L}} \mathcal{L}'(p')$.

FULFILLMENT OF B-REQUESTS. Consider a point $p = (x, y)$ in $\mathcal{G}'$ and some B-request

$\alpha \in \mathcal{R}eq_{\mathrm{B}}(\mathcal{L}'(p))$ associated with it. Since, by construction, $\alpha \in \mathcal{R}eq_{\mathrm{B}}(\mathcal{L}(f(p)))$ and $\mathcal{G}$ is a (fulfilling) compass structure, we know that $\mathcal{G}$ contains a point $q' = (x', y')$ such that $f(p)\ \mathrm{B}\ q'$ and $\alpha \in \mathcal{O}bs(\mathcal{L}(q'))$. We prove that $\mathcal{G}'$ contains a point p' such that $p\ \mathrm{B}\ p'$ and $\alpha \in \mathcal{O}bs(\mathcal{L}'(p'))$ by distinguishing among the following three cases:

(1) $y < y_0$,

(2) $y' \geqslant y_1$,

(3) $y \geqslant y_0$ and $y' < y_1$.

If $y < y_0$, then, by construction, we have $p = f(p)$ and $q' = f(q')$. Therefore, we simply define $p' = q'$ in such a way that $p = f(p)\ \mathrm{B}\ q' = p'$ and $\alpha \in \mathcal{O}bs(\mathcal{L}'(p'))$ $(= \mathcal{O}bs(\mathcal{L}(f(p'))) = \mathcal{O}bs(\mathcal{L}(q')))$.

If $y' \geqslant y_1$, then, by construction, we have either $f(p) = (f(x), y + k)$ or $f(p) = (x + k, y+k)$, depending on whether $x < y_0$ or $x \geqslant y_0$. We define $p' = (x, y'-k)$ in such a way that $p\ \mathrm{B}\ p'$. Moreover, we observe that either $f(p') = (f(x), y')$ or $f(p') = (x + k, y')$, depending on whether $x < y_0$ or $x \geqslant y_0$, and in both cases $f(p') = q'$ follows. This shows that $\alpha \in \mathcal{O}bs(\mathcal{L}'(p'))$ $(= \mathcal{O}bs(\mathcal{L}(f(p'))) = \mathcal{O}bs(\mathcal{L}(q')))$.

If $y \geqslant y_0$ and $y' < y_1$, then we define $\bar{p} = (x, y_0)$ and $\bar{q} = (x', y_1)$ and we observe that $f(p)\ \mathrm{B}\ \bar{q}$, $\bar{q}\ \mathrm{B}\ q'$, and $f(\bar{p}) = \bar{q}$. From $f(p)\ \mathrm{B}\ \bar{q}$ and $\bar{q}\ \mathrm{B}\ q'$, it follows that $\alpha \in \mathcal{R}eq_{\mathrm{B}}(\mathcal{L}(\bar{q}))$ and hence $\alpha \in \mathcal{R}eq_{\mathrm{B}}(\mathcal{L}(\bar{p}))$. Since $\mathcal{G}$ is a (fulfilling) compass structure, we know that there is a point p' such that $\bar{p}\ \mathrm{B}\ p'$ and $\alpha \in \mathcal{O}bs(\mathcal{L}(\bar{p}'))$. Moreover, since $\bar{p}\ \mathrm{B}\ p'$, we have $f(p') = p'$, from which we obtain $p\ \mathrm{B}\ p'$ and $\alpha \in \mathcal{O}bs(\mathcal{L}(p'))$.

FULFILLMENT OF $\bar{\mathrm{B}}$-REQUESTS. The proof is symmetric to the previous one.

FULFILLMENT OF A-REQUESTS. Consider a point $p = (x, y)$ in $\mathcal{G}'$ and some A-request $\alpha \in \mathcal{R}eq_{\mathrm{A}}(\mathcal{L}'(p))$ associated with p in $\mathcal{G}'$. Since, by previous arguments, $\mathcal{G}'$ fulfills all $\bar{\mathrm{B}}$-requests of its atoms, it is sufficient to prove that either $\alpha \in \mathcal{O}bs(\mathcal{L}'(p'))$ or $\alpha \in \mathcal{R}eq_{\bar{\mathrm{B}}}(\mathcal{L}'(p'))$, where $p' = (y, y + 1)$. This can be easily proved by distinguishing among the three cases $y < y_0 - 1$, $y = y_0 - 1$, and $y \geqslant y_0$.

FULFILLMENT OF $\bar{\mathrm{L}}$-REQUESTS. Consider a point $p = (x, y)$ in $\mathcal{G}'$ and some $\bar{\mathrm{L}}$-request $\alpha \in \mathcal{R}eq_{\bar{\mathrm{L}}}(\mathcal{L}'(p))$ associated with it. Since, by construction, $\alpha \in \mathcal{R}eq_{\bar{\mathrm{L}}}(\mathcal{L}(f(p)))$ and $\mathcal{G}$ is a (fulfilling) compass structure, we know that $\mathcal{G}$ contains a point $q' = (x', y')$ such that $f(p)\ \bar{\mathrm{L}}\ q'$ and $\alpha \in \mathcal{O}bs(\mathcal{L}(q'))$. To simplify the proofs, we assume that q' is *minimal* with respect to the vertical coordinate, that is, for every other point $q'' = (x'', y'')$ with $y'' < y'$, $\alpha \notin \mathcal{O}bs(\mathcal{L}(q''))$. We prove that $\mathcal{G}'$ contains a point p' such that $p\ \bar{\mathrm{L}}\ p'$ and $\alpha \in \mathcal{O}bs(\mathcal{L}'(p'))$ by distinguishing among the following five cases (notice that exactly one of such cases holds):

(1) $y \leqslant y_0$,

(2) $x < y_0$ and $y \geqslant y_0$,

(3) $x \geqslant y_0$ and $y' < y_1$,

(4) $x \geqslant y_0$ and $y' = y_1$,

(5) $x \geqslant y_0$ and $y' > y_1$.

If $y < y_0$, then, by construction, we have $p = f(p)$ and $q' = f(q')$. Therefore, we simply define $p' = q'$ in such a way that $p = f(p) \bar{L} q' = p'$ and $\alpha \in \mathcal{O}bs(\mathcal{L}'(p'))$ $(= \mathcal{O}bs(\mathcal{L}(f(p'))) = \mathcal{O}bs(\mathcal{L}(q')))$.

If $x < y_0$ and $y \geqslant y_0$ then $f(p) = (f(x), y + k)$. Now, consider the point $p'' = (f(x), y_1)$: since $f(p)Bp''$ and $\mathcal{G}$ is a consistent compass structure, we have that $\mathcal{R}eq_{\bar{L}}(p'') = \mathcal{R}eq_{\bar{L}}(f(p))$. By definition of f, we have that $\mathcal{L}(f(x), y_1) = \mathcal{L}(x, y_0)$ and thus, since $\mathcal{G}$ is fulfilling, there exists a point $p' = (x'', y'')$ such that $y'' < x$ and $\alpha \in \mathcal{O}bs(\mathcal{L}(p'))$. Since $f(p') = p'$, this shows that $\alpha \in \mathcal{O}bs(\mathcal{L}'(p'))$ as well.

If $x \geqslant y_0$ and $y' < y_1$ then $f(p) = (x + k, y + k)$. Since $\mathcal{G}$ is a consistent compass structure, we have that $\alpha \in \mathcal{R}eq_{\bar{L}}(\mathcal{L}(y_1 - 1, y_1))$. By the definition of compatible rows, we have that $\mathcal{L}(y_1 - 1, y_1) = \mathcal{L}(y_0 - 1, y_0)$ and thus (by the minimality assumption) $y' < y_0$ and $q' = f(q')$. Therefore, we simply define $p' = q'$ in such a way that $p \bar{L} q' = p'$ and $\alpha \in \mathcal{O}bs(\mathcal{L}'(p'))$ $(= \mathcal{O}bs(\mathcal{L}(f(p'))) = \mathcal{O}bs(\mathcal{L}(q')))$.

If $x \geqslant y_0$ and $y' = y_1$ then $\mathcal{L}(q') \in \mathcal{S}ha_{\mathcal{G}}(y_1)$. By the definition of compatible rows, we have that $\mathcal{S}ha_{\mathcal{G}}(y_1) = \mathcal{S}ha_{\mathcal{G}}(y_0)$ and thus there must exists a point $q'' = (x'', y_0)$ such that $\mathcal{L}(q') = \mathcal{L}(q'')$ and $y_0 < y'$, against the hypothesis that q' is a minimal point satisfying α. Hence, this case cannot happen.

If $x \geqslant y_0$ and $y' > y_1$ then, by the minimality assumption on q' we have that for every $y'' < y'$, $\alpha \notin \mathcal{O}bs(\mathcal{L}(x'', y''))$ for any $x'' < y''$. Hence, by the definition of witness set, we have that there exists a witness $(x_\alpha, y_\alpha) \in \mathcal{W}it(y_1)$ such that $\alpha \in \mathcal{O}bs(\mathcal{L}(x_\alpha, y_\alpha))$ and $y_\alpha = y'$ (by the minimality assumption). If $x_\alpha \geqslant y_1$ then we define $p' = (x_\alpha - k, y_\alpha - k)$. Otherwise, $x_\alpha < y_1$ and by the definition of the mapping function w and of the function f, we have that $f(w(x_\alpha)) = x_\alpha$: we define $p' = (w(x_\alpha), y' - k)$. In both cases we have that $f(p') = (x_\alpha, y_\alpha)$, $p\bar{L}p'$ and $\alpha \in \mathcal{O}bs(\mathcal{L}'(p'))$.

FEATURED FORMULAS. Recall that $\varphi \in \mathcal{L}(0, 1)$. Since our contraction procedure never changes the labelling of the initial point, $\varphi \in \mathcal{L}'(0, 1)$ as well. □

On the grounds of the above result, we can provide a suitable upper bound for the length of a minimal finite interval structure that satisfies φ, if there exists any. This yields a straightforward, but inefficient, 2NEXPTIME algorithm that decides whether a given $AB\bar{B}\bar{L}$-formula φ is satisfiable over finite interval structures.

Theorem 6. *An $AB\bar{B}\bar{L}$-formula φ is satisfied by some finite interval structure iff it is featured by some compass structure of length $N \leqslant (8|\varphi|+15)^{2^{36|\varphi|+63}} \cdot 2^{36|\varphi|+63}$ (i.e., double exponential in $|\varphi|$).*

Proof. Suppose that φ is satisfied by a finite interval structure $\mathcal{S}$, and let $\xi = \varphi \vee \langle\bar{B}\rangle\varphi \vee \langle A\rangle\varphi \vee \langle A\rangle\langle A\rangle\varphi$. By Proposition 2, there is a compass structure $\mathcal{G}$ that features ξ on the initial point and has some finite size N. By Lemma 5, we can assume without loss of generality that all rows of $\mathcal{G}$ are pairwise incompatible. Recall from Section 2.3 that $\mathcal{G}$ contains at most $2^{9|\xi|}$ distinct atoms. For every row y of the compass structure and every atom $F \in \mathcal{A}_\xi$, let $\#(F, y)$ be the cardinality of the set $\{(x, y) : x < y \text{ and } \mathcal{L}(x, y) = F\}$. We associate to every row y of the structure a *characteristic function* $c_y : \mathcal{A}_\xi \mapsto \mathbb{N}$ defined as follows:

$$c_y(F) = \begin{cases} \#(F, y) & \#(F, y) \leqslant 2|\xi| \\ 2|\xi| & \text{otherwise} \end{cases} \tag{1}$$

Since any witness set $\mathcal{W}it(y)$ contains at most $2|\xi|$ witnesses, it is easy to see that two rows y_0 and y_1 with the same characteristic function and such that $\mathcal{L}(y_0-1, y_0) = \mathcal{L}(y_1-1, y_1)$ are compatible. The number of possible characteristic functions is bounded by $(2|\xi| + 1)^{2^{9|\xi|}}$, and thus $\mathcal{G}$ cannot have more than $(2|\xi| + 1)^{2^{9|\xi|}} \cdot 2^{9|\xi|}$ rows. Since $|\xi| = 4|\varphi| + 7$ we have that $N \leqslant (8|\varphi| + 15)^{2^{36|\varphi|+63}} \cdot 2^{36|\varphi|+63}$, that is, double exponential in $|\varphi|$. $\square$

3.2. *A small-model theorem for infinite structures*

In general, compass structures may be infinite. Here, we prove that we can restrict our attention to sufficiently "regular" infinite compass structures, which can be represented in double exponential space with respect to $|\varphi|$. To do that, we introduce the notion of compass structure generator, which is a finite compass structure that can be extended to an infinite fulfilling one.

Definition 7. *We say that a finite compass structure* $\mathcal{G} = (\mathbb{P}_{\mathbb{O}}, \mathcal{L})$ *of size* N *is* partially fulfilling *if for every point* $(x, y) \in \mathbb{P}_{\mathbb{O}}$ *such that* $y < N - 1$*, for every relation* $R \in \{A, B, \bar{B}, \bar{L}\}$*, and, for every formula* $\psi \in \mathcal{R}eq_R(\mathcal{L}(p))$*, one of the following conditions hold:*

1. *there exists a point* $p' \in \mathbb{P}_{\mathbb{O}}$ *such that* $p \; R \; p'$ *and* $\psi \in \mathcal{O}bs(\mathcal{L}(p'))$ *(ψ is fulfilled in p'),*
2. $R = \bar{B}$ *and* $\psi \in \mathcal{R}eq_{\bar{B}}(\mathcal{L}(x, N-1))$*,*
3. $R = A$ *and* $\psi \in \mathcal{R}eq_{\bar{B}}(\mathcal{L}(y, N-1))$*,*
4. $R = \bar{L}$ *and* $\psi \in \mathcal{R}eq_{\bar{L}}(\mathcal{L}(0, 1))$*.*

Notice that all B-requests are fulfilled in a partially fulfilling compass structure and that $\bar{B}$ (resp., A, $\bar{L}$) requests are either fulfilled or "transferred to the border" of the compass structure. Moreover, any substructure $\mathcal{G}'$ of a fulfilling compass structure $\mathcal{G}$ is partially fulfilling.

Definition 8. *Given a finite compass structure* $\mathcal{G} = (\mathbb{P}_{\mathbb{O}}, \mathcal{L})$ *and a row* y*, a* future witness set *for* y *is any* minimal *set* $\mathcal{F}ut\mathcal{W}it(y) \subseteq \{x : x < y\}$ *such that for every* $F \in \mathcal{S}ha_{\mathcal{G}}(y)$ *there exists a witness* $x_F \in \mathcal{F}ut\mathcal{W}it(y)$ *that respects the following properties:*

1. $\mathcal{L}(x_F, y) = F$*,*
2. *for every* $\psi \in \mathcal{R}eq_{\bar{B}}(F)$ *there exists a point* $(x_F, y') \in \mathcal{G}$ *with* $y' > y$ *and* $\psi \in \mathcal{O}bs(\mathcal{L}(x_F, y))$*.*

Since $\mathcal{F}ut\mathcal{W}it(y)$ is minimal, we have that for every $F \in \mathcal{S}ha(y)$ there is exactly one witness x_F in $\mathcal{F}ut\mathcal{W}it(y)$. Hence, $|\mathcal{F}ut\mathcal{W}it(y))| \leqslant 2^{9|\varphi|}$.

Definition 9. *Given a finite compass structure* $\mathcal{G} = (\mathbb{P}_{\mathbb{O}}, \mathcal{L})$ *and a row* y*, a* past witness set for y *is any* minimal *set* $\mathcal{P}ast\mathcal{W}it(y) \subseteq \mathbb{P}_{\mathbb{O}}$ *such that for every request* $\psi \in \mathcal{R}eq_{\bar{L}}(\mathcal{O}bs(\mathcal{L}(y-1, y))$ *there exists a witness* (x_ψ, y_ψ) *such that* $\psi \in \mathcal{O}bs(\mathcal{L}(x_\psi, y_\psi))$ *and* $y_\psi < y - 1$.

Again, by the minimality of $\mathcal{P}ast\mathcal{W}it(y)$ we have that there is at most one distinct point for every $\bar{L}$-formula in $\mathcal{L}(y-1, y)$ and thus $|\mathcal{P}ast\mathcal{W}it(y)| \leqslant |\mathcal{R}eq_{\bar{L}}(y-1, y)| \leqslant |\mathcal{C}l(\varphi)| \leqslant 2 \cdot |\varphi|$.

We concentrate our attention on infinite structures that are unbounded both on the future and on the past (i.e., based on the set of integers $\mathbb{Z}$). The case when the structure is unbounded only in one direction (e.g., the naturals $\mathbb{N}$ or the set of negative integers $\mathbb{Z}^-$) can be dealt with in a similar way, by appropriately adapting the following notions.

Definition 10. *Given an* $\mathsf{AB\bar{B}\bar{L}}$ *formula* φ *and a finite, partially fulfilling compass structure* $\mathcal{G} = (\mathbb{P}_{\mathbb{O}}, \mathcal{L})$ *of size* N*, we say that* $\mathcal{G}$ *is a* compass generator for φ *if there exist four rows* y_φ*,* y_0*,* y_1*, and* y_2 *which satisfy the following properties:*

G1 $y_0 < y_1 < y_2$ *and* $y_0 \leqslant y_\varphi$*,*

G2 $\varphi \in \mathcal{L}(y_\varphi - 1, y_\varphi)$ *or* $\langle \bar{B} \rangle \varphi \in \mathcal{L}(y_\varphi - 1, y_\varphi)$*,*

G3 $\mathcal{S}ha(y_1) \subseteq \mathcal{S}ha(y_0)$ *and* $\mathcal{L}(y_0 - 1, y_0) = \mathcal{L}(y_1 - 1, y_1)$*,*

G4 *there exists a past witness set* $\mathcal{P}ast\mathcal{W}it(y_1)$ *such that* $y_0 \leqslant \min(\pi_{y_1}(\mathcal{P}ast\mathcal{W}it(y_1)))$*,*

G5 $\mathcal{S}ha(\mathsf{N}-1) \subseteq \mathcal{S}ha(y_2)$ *and* $\mathcal{L}(y_2 - 1, y_2) = \mathcal{L}(\mathsf{N}-2, \mathsf{N}-1)$*,*

G6 *there exists a future witness set* $\mathcal{F}ut\mathcal{W}it(y_2)$ *for* y_2.

The next theorem shows that the information contained in a compass generator for φ is sufficient to build an infinite fulfilling compass structure featuring φ.

Theorem 11. *An* $\mathsf{AB\bar{B}\bar{L}}$ *formula* φ *is satisfiable over the integers* $\mathbb{Z}$ *if and only if there exists a compass generator* $\mathcal{G} = (\mathbb{P}_{\mathbb{O}}, \mathcal{L})$ *for* φ.

Proof. **(⇒)** Let φ an $\mathsf{AB\bar{B}\bar{L}}$ formula that is satisfiable over an infinite fulfilling compass structure $\mathcal{G} = (\mathbb{P}_{\mathbb{Z}}, \mathcal{L})$. Since $\mathcal{G}$ features φ we have that there exists a point (x, y) with $\varphi \in \mathcal{L}(x, y)$ and thus the row $y_\varphi = x + 1$ respects condition **G2**.

Now, let $\mathcal{I}nf(\mathcal{G})$ be the set of shadings that occurs infinitely often in $\mathcal{G}$. We define y_1 as the greatest row such that for every $y' \leqslant y_1$, $\mathcal{S}ha(y') \in \mathcal{I}nf(\mathcal{G})$, and y_2 as the smallest row such that for every $y' \geqslant y_2$, $\mathcal{S}ha(y') \in \mathcal{I}nf(\mathcal{G})$. Clearly, since $\mathcal{G}$ is unbounded in the past, we can find two rows y_{min} and y_0 such that $y_{min} < y_0$, and a corresponding portion of the grid $\mathbb{P}_{y_{min}} = \{(x, y) : x \geqslant y_{min}\}$ such that *(i)* $y_0 \leqslant y_\varphi$, *(ii)* $y_0 < y_1$, *(iii)* $\mathcal{S}ha(y_1) \subseteq \mathcal{S}ha(y_0)$ in $\mathbb{P}_{y_{min}}$, *(iv)* $\mathcal{L}(y_0 - 1, y_0) = \mathcal{L}(y_1 - 1, y_1)$, and *(v)* there exists a past witness set $\mathcal{P}ast\mathcal{W}it(y_1)$ for y_1 such that $y_0 \leqslant \min(\pi_{y_1}(\mathcal{P}ast\mathcal{W}it(y_1)))$ in $\mathbb{P}_{y_{min}}$. Hence, conditions **G3** and **G4** are respected.

Symmetrically, since $\mathcal{G}$ is unbounded in the future, we can find a row $y_{max} > y_2$ and a corresponding portion of the grid $\mathbb{P}_{y_{min}}^{y_{max}} = \{(x, y) : x \geqslant y_{min} \wedge y \leqslant y_{max}\}$ such that

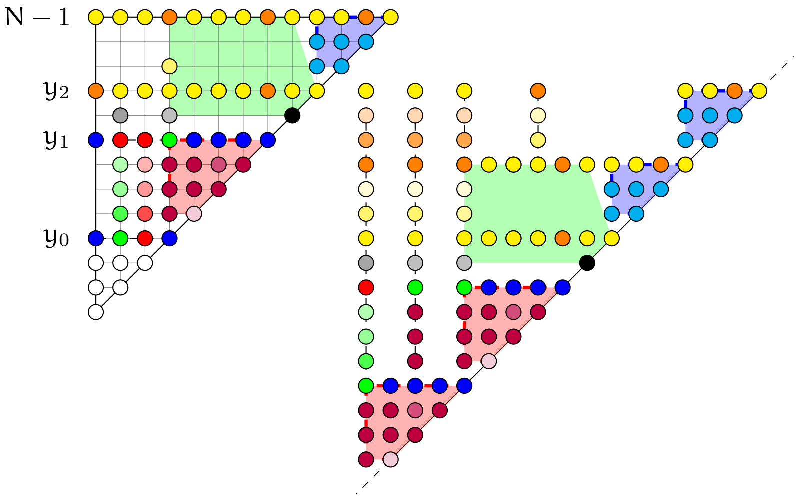

Figure 4. A compass generator (left) and a portion of the generated infinite compass structure (right).

(1) $\mathcal{S}ha(y_{max}) \subseteq \mathcal{S}ha(y_2)$,
(2) $\mathcal{L}(y_2 - 1, y_2) = \mathcal{L}(y_{max} - 1, y_{max})$, and
(3) there exists a future witness set $\mathcal{F}ut\mathcal{W}it(y_2)$ for y_2 in $\mathbb{P}_{y_{min}}^{y_{max}}$.

This shows that conditions **G5** and **G6** are respected as well. Since $y_0 \leqslant y_\varphi$ and $y_0 < y_1 < y_2$ condition **G1** is also respected. Since the restriction of $\mathcal{G}$ to the finite grid $\mathbb{P}_{y_{min}}^{y_{max}}$ is a partially fulfilling compass structure, we have found the required compass generator for φ.

(⇐) Let $\mathcal{G} = (\mathbb{P}_{\mathbb{O}}, \mathcal{L})$ be a compass generator of size N for φ and let $y_0 < y_1 < y_2$ and y_φ be the four rows that satisfy properties **G1–G6** of Definition 10. We will define an infinite sequence of partially fulfilling compass structures $\mathcal{G}_0 \subset \mathcal{G}_1 \subset \mathcal{G}_2 \subset \dots$ such that the infinite union $\mathcal{G}^\omega = \bigcup_{i=0}^{+\infty} \mathcal{G}_i$ is an infinite fulfilling compass structure that features φ. We start from the initial compass structure $\mathcal{G}_0 = (\mathbb{P}^0, \mathcal{L}^0)$ where $\mathbb{P}^0 = \{(x, y) \in \mathbb{P}_{\mathbb{O}} : x \geqslant y_0 - 1 \wedge y_0 \leqslant y < N\}$ and $\mathcal{L}^0(x, y) = \mathcal{L}(x, y)$ for every point $(x, y) \in \mathbb{P}_{\mathbb{O}}^0$, and we will show how to iteratively build the infinite sequence of compass structures. For every step i of the procedure, let $\mathcal{G}_i = (\mathbb{P}^i, \mathcal{L}^i)$ be the current structure, and let y_{min}^i and y_{max}^i be the minimum and maximum vertical coordinate in $\mathbb{P}^i$, respectively. We guarantee that the following invariant is respected:

(INV) $\mathcal{S}ha_{\mathcal{G}_i}(y_{max}^i) \subseteq \mathcal{S}ha_{\mathcal{G}}(y_2)$,
$\mathcal{S}ha_{\mathcal{G}_i}(y_{min}^i + y_1 - y_0) \subseteq \mathcal{S}ha_{\mathcal{G}}(y_0)$,
$\mathcal{L}^i(y_{max}^i - 1, y_{max}^i) = \mathcal{L}(y_2 - 1, y_2)$, and $\mathcal{L}^i(y_{min}^i - 1, y_{min}^i) = \mathcal{L}(y_0 - 1, y_0)$.

The invariant trivially holds for $\mathcal{G}_0$. Now, suppose that $\mathcal{G}_i$ respects **(INV)** and let $k_{past} = y_1 - y_0$ and $k_{future} = N - y_2$. Figure 4 depicts how $\mathcal{G}_{i+1} = (\mathbb{P}^{i+1}, \mathcal{L}^{i+1})$ can be built from $\mathcal{G}_i$. Formally, the procedure is defined as follows.

a) $y^{i+1}_{min} = y^i_{min} - k_{past}$, $y^{i+1}_{max} = y^i_{max} + k_{future}$, and $\mathbb{P}^{i+1} = \{(x, y) \in \mathbb{P}_{\mathbb{Z}} : x \geqslant y^{i+1}_{min} - 1 \wedge y^{i+1}_{min} \leqslant y < y^{i+1}_{max}\}$.
b) for every point $p \in \mathbb{P}^{i+1} \cap \mathbb{P}^i$, let $\mathcal{L}^{i+1}(p) = \mathcal{L}^i(p)$.
c) for every point $(x, y) \in \mathbb{P}^{i+1} \setminus \mathbb{P}^i$ such that $y \leqslant y^i_{min}$, let $\mathcal{L}^{i+1}(x, y) = \mathcal{L}^i(x + k_{past}, y + k_{past})$ (red area in Fig. 4).
d) for every point $(x, y) \in \mathbb{P}^{i+1} \setminus \mathbb{P}^i$ such that $x \geqslant y^i_{max}$, let $\mathcal{L}^{i+1}(x, y) = \mathcal{L}^i(x - k_{future}, y - k_{future})$ (blue area in Fig. 4).
e) By construction, for every point (x, y^i_{min}) with $x < y^i_{min} - 1$ we have that $\mathcal{L}^{i+1}(x, y^i_{min}) = \mathcal{L}^i(x + k_{past}, y^i_{min} + k_{past})$. Since $\mathcal{G}_i$ respects the invariant, $\mathcal{L}^i(x + k_{past}, y^i_{min} + k_{past}) = \mathcal{L}^{i+1}(x, y^i_{min}) \in \mathcal{S}ha_{\mathcal{G}}(y_0)$. Let $(\bar{x}, y_0)$ be a point on the row y_0 with the same labelling of $\mathcal{L}^{i+1}(x, y^i_{min})$: we define the labelling of all points $(x, y^i_{min} + j)$, with $1 \leqslant j \leqslant k_{past}$, as $\mathcal{L}^{i+1}(x, y^i_{min} + j) = \mathcal{L}(\bar{x}, y_0 + j)$. Now, since $\mathcal{L}(\bar{x}, y_0 + k_{past}) \in \mathcal{S}ha_{\mathcal{G}}(y_1)$ ($y_1 = y_0 + k_{past}$) and $\mathcal{S}ha_{\mathcal{G}}(y_1) \subseteq \mathcal{S}ha_{\mathcal{G}}(y_0)$ (**G3**), we can find a point $(\hat{x}, y_0)$ on the row y_0 with the same labelling of $\mathcal{L}^{i+1}(x, y^i_{min} + k_{past})$ and define the labelling of every point $(x, y^i_{min} + k_{past} \cdot j)$ for every $1 < j \leqslant i + 1$. At the end of this procedure we have labelled all points (x, y) such that $y \leqslant y_1$.
f) For every point (x, y_1), by construction, we have that $\mathcal{L}^{i+1}(x, y_1) \in \mathcal{S}ha_{\mathcal{G}}(y_1)$. Let $(\bar{x}, y_1)$ be a point such that $\mathcal{L}^{i+1}(x, y_1) = \mathcal{L}(\bar{x}, y_1)$. As in the previous case, we define the labelling of all points (x, y), with $y_1 < y \leqslant y_2$ as $\mathcal{L}^{i+1}(x, y) = \mathcal{L}(\bar{x}, y)$. At the end of this step we labelled all points (x, y) such that $y \leqslant y_2$.
g) Now, by construction, for every point (x, y_2) we have that $\mathcal{L}^{i+1}(x, y_2) \in \mathcal{S}ha_{\mathcal{G}}(y_2)$. By condition **G6** of Definition 10, there exists a point $\bar{x} \in \mathcal{F}ut\mathcal{W}it(y_2)$ such that $\mathcal{L}^{i+1}(x, y_2) = \mathcal{L}(\bar{x}, y_2)$. We define $\mathcal{L}^{i+1}(x, y_2 + j) = \mathcal{L}(\bar{x}, y_2 + j)$ for every $1 \leqslant j \leqslant k_{future}$. Since $y_2 + k_{future} = N - 1$ and $\mathcal{S}ha(N - 1) \subseteq \mathcal{S}ha(y_2)$ we have that $\mathcal{L}^{i+1}(x, N - 1) \in \mathcal{S}ha(y_2)$ (**G5**) and thus we can repeat this procedure iteratively until we have labelled all points (x, y) such that $y \leqslant y^{i+1}_{max}$ and $x < y^i_{min} - 1$.
h) To conclude the procedure, we must define the labelling of points (x, y) such that $x \geqslant y^i_{min} - 1$ and $y \geqslant y^i_{max}$. Note that for every point (x, y^i_{max}) with $x \geqslant y^i_{min} - 1$ we have, by the invariant, that $\mathcal{S}ha_{\mathcal{G}^i}(y^i_{max}) \subseteq \mathcal{S}ha_{\mathcal{G}}(y_2)$. Then there exists a point $\bar{x} \in \mathcal{F}ut\mathcal{W}it(y_2)$ such that $\mathcal{L}^{i+1}(x, y^i_{max}) = \mathcal{L}(\bar{x}, y_2)$. We define $\mathcal{L}^{i+1}(x, y^i_{max} + j) = \mathcal{L}(\bar{x}, y_2 + j)$ for every $1 \leqslant j \leqslant k_{future}$.

It is easy to see that $\mathcal{G}_i$ is a partially fulfilling compass structure that respects the invariant. Moreover, suppose that for some point $p = (x, y) \in \mathbb{P}^i$ and relation $R \in \{A, B, \bar{B}, \bar{L}\}$ there exists $\alpha \in \mathcal{R}eq_R(p)$ that is not fulfilled in $\mathcal{G}_i$. We show that $\mathcal{G}_{i+1}$ fulfills the R-request α for p.

- If $R = A$, since $\mathcal{G}^i$ is partial fulfilling and it is finite we have that the point $p' = (y, y^i_{max})$ is such that $\alpha \in \mathcal{R}eq_{\bar{B}}(\mathcal{L}(p'))$. By step h) of the procedure, and by the definition of future witness set, $\mathcal{G}_{i+1}$ contains a point $p'' = (y, y^i_{max} + j)$ such that $\alpha \in \mathcal{L}^{i+1}(p'')$.
- If $R = B$, by Definition 7 we have that all the B-requests in a partial fulfilling

compass structure are fulfilled and thus this case connot be given.

- If $R = \bar{B}$ the case is analogous to the case of $R = A$.
- If $R = \bar{L}$, since $\mathcal{G}^i$ is partial fulfilling and it is finite we have that $\alpha \in \mathcal{R}eq_{\bar{L}}(\mathcal{L}(y^i_{min} - 1, y^i_{min}))$. By point c) of the construction we have that $\mathcal{L}^i(y^i_{min} - 1, y^i_{min}) = \mathcal{L}(y_0 - 1, y_0) = \mathcal{L}(y_1 - 1, y_1)$. Hence, by condition **G4** of Definition 10 and by the definition of past witness set, there exists a point $(\bar{x}, \bar{y})$ with $y_0 \leqslant \bar{x} < \bar{y} \leqslant y_1$ such that $\alpha \in \mathcal{L}(\bar{x}, \bar{y})$. By construction we have that $\mathcal{L}(\bar{x}, \bar{y}) = \mathcal{L}^{i+1}(\bar{x} - (i+1) \cdot k_{past}, \bar{y} - (i+1) \cdot k_{past})$ and thus and thus the $\bar{L}$-request α for the point p is fulfilled at step $i+1$ by the point $(\bar{x} - (i+1) \cdot k_{past}, \bar{y} - (i+1) \cdot k_{past})$.

Hence, we can conclude that the infinite compass structure $\mathcal{G}^\omega$ is fulfilling. By condition **G2** of Definition 10 we have that $\mathcal{G}^\omega$ features φ and thus that φ is satisfiable over the integers. □

Theorem 11 shows that satisfiability of a formula over infinite models can be reduced to the existence of a finite compass generator for it. However, it does not give any bound on the size of it. In the following we will show how the techniques exploited in Section 3.1 for finite models can be adapted to obtain a doubly exponential bound on the size of compass generators.

Definition 12. *Given a compass generator $\mathcal{G} = (\mathbb{P}_{\mathbb{O}}, \mathcal{L})$, we say that two rows $y < y'$ are* globally compatible *if and only if the following properties holds:*

(1) $\mathcal{L}(y-1, y) = \mathcal{L}(y'-1, y')$ *and* $\mathcal{S}ha_{\mathcal{G}}(y) = \mathcal{S}ha_{\mathcal{G}}(y')$,

(2) for every $\bar{y} \in \{y_\varphi, y_0, y_1, y_2\}$ *it is not the case that* $y \leqslant \bar{y} \leqslant y'$,

(3) there exists a past witness set $\mathcal{P}ast\mathcal{W}it(y_1)$ *such that for every point* $(\bar{x}, \bar{y}) \in \mathcal{P}ast\mathcal{W}it(y_1)$ *it is not the case that* $y \leqslant \bar{y} \leqslant y'$,

(4) there exists a future witness set $\mathcal{F}ut\mathcal{W}it(y_2)$ *such that for every point* $\bar{x} \in \mathcal{F}ut\mathcal{W}it(y_2)$ *and every* $\bar{B}$*-request* $\alpha \in \mathcal{R}eq_{\bar{B}}(\mathcal{L}(\bar{x}, y_2)$ *there is a point* $(\bar{x}, \bar{y})$ *such that* $y_2 < \bar{y}$, $\alpha \in \mathcal{O}bs(\mathcal{L}(\bar{x}, y_2))$ *and it is not the case that* $y \leqslant \bar{y} \leqslant y'$,

(5) there exists a witness set $\mathcal{W}it(y')$ *for* y' *and an* injective mapping function $w : \pi_{y'}(\mathcal{W}it(y') \cup \mathcal{P}ast\mathcal{W}it(y_1) \cup \mathcal{F}ut\mathcal{W}it(y_2)) \mapsto \{x : x < y\}$, *such that* $\mathcal{L}(x, y') = \mathcal{L}(w(x), y)$, *for every* $x \in \pi_{y'}(\mathcal{W}it(y') \cup \mathcal{P}ast\mathcal{W}it(y_1) \cup \mathcal{F}ut\mathcal{W}it(y_2))$, *and* $w(x) = x$, *for every* $x \in \pi_{y'}(\mathcal{P}ast\mathcal{W}it(y_1)$.

Clearly, two globally compatible rows are compatible. The additional conditions of the definition guarantees that the contraction procedure do not remove "meaningful" parts of the compass generator, such as the rows y_φ, y_0, y_1, and y_2 (condition 2) or future and past witnesses (conditions 3 and 4).

Lemma 13. *Let $\mathcal{G}$ be a compass generator for φ of size N. If there exist two global-compatible rows $0 < y < y' < N$ in $\mathcal{G}$, then there exists a compass generator $\mathcal{G}'$ of size $N' = N - y + y'$ that features φ.*

Proof. We can define a function $f : \{0, ..., y\} \rightarrow \{0, ..., y'\}$ and contract $\mathcal{G}$ to a smaller compass structure $\mathcal{G}'$ in the very same way of Lemma 5. It can be easily proved that the obtained $\mathcal{G}'$ is a partial fulfilling compass structure. Let $k = y' - y$ and let $y'_\varphi = y_\varphi$ if $y_\varphi < y$, $y'_\varphi = y_\varphi - k$ otherwise. To prove that $\mathcal{G}'$ is a compass generator, let us consider the following four cases.

- If $y' < y_0$, then we have that $y'_i = y_i - k$ for $i \in \{0, 1, 2, \varphi\}$ satisfy conditions **G1-G6** in $\mathcal{G}'$.
- If $y_0 < y < y' < y_1$, then for every point $(\bar{x}, \bar{y}) \in \mathcal{P}ast\mathcal{W}it(y_1)$ we have that either $f(\bar{x}, \bar{y}) = (\bar{x}, \bar{y})$ (when $\bar{y} < y$) or $f(\bar{x}, \bar{y} - k) = (w(\bar{x}), \bar{y} - k) = (\bar{x}, \bar{y})$ (when $\bar{y} > y'$), and thus $\mathcal{P}ast\mathcal{W}it(y_1)$ is a past witness set for $\mathcal{G}'$ as well. From this we can conclude that $y'_\varphi, y_0, y_1 - k$, and $y_2 - k$ satisfy conditions **G1-G6** in $\mathcal{G}'$.
- If $y_0 < y_1 < y < y' < y_2$, then it is easy to prove that y'_φ, y_0, y_1 and $y_2 - k$ satisfy **G1-G6** in $\mathcal{G}'$.
- If $y_0 < y_1 < y_2 < y < y'$, then it is easy to observe that y'_φ, y_0, y_1 and y_2 satisfy **G1-G6** in $\mathcal{G}'$.

Hence, in all possible cases $\mathcal{G}'$ is a compass generator for φ. □

Theorem 14. *An* $AB\bar{B}\bar{L}$*-formula* φ *is satisfied by some infinite interval structure iff it is featured by some compass generator of length* $N \leqslant (2|\varphi| + 1)^{2^{9|\varphi|}} \cdot 2^{18|\varphi|^2+9|\varphi|}$ *(i.e., double exponential in* $|\varphi|$*).*

Proof. Suppose that φ is satisfied by a infinite interval structure $\mathcal{S}$. By Theorem 11, there is a compass generator $\mathcal{G}$ that features φ. By Lemma 13, we can assume without loss of generality that all rows of $\mathcal{G}$ are pairwise global-incompatible. Let c_y the characteristic function defined in the proof of Theorem 6. Now, let $x_1 < \ldots < x_k$ be the ordered sequence of the points in $\mathcal{P}ast\mathcal{W}it(y_1)$. We associate to every row y a finite word W_y of length $|W_y| \leqslant k \leqslant 2 \cdot |\varphi|$ on the alphabet $\mathcal{A}_\varphi$ ($|\mathcal{A}_\varphi| = 2^{9|\varphi|}$) such that for every $x_i \in \mathcal{P}ast\mathcal{W}it(y_1)$, $W(i) = \mathcal{L}(x_i, y)$. It is easy to prove that two rows $y < y'$ in $\mathbb{O}$ with $c_y(F) = c_{y'}$, $W_y = W_{y'}$ and such that $\mathcal{L}(y'-1, y') = \mathcal{L}(y-1, y)$ are global-compatible.

Since the number of possible characteristic functions is bounded by $(2|\varphi|+1)^{2^{9|\varphi|}}$, and the number of possible words is bounded by $(2^{9|\varphi|})^{2\cdot|\varphi|} = 2^{18|\varphi|^2}$, $\mathcal{G}$ cannot have more than $(2|\varphi| + 1)^{2^{9|\varphi|}} \cdot 2^{18|\varphi|^2+9|\varphi|}$ rows, and thus N is at most doubly exponential in $|\varphi|$.□

4. Complexity bounds to the satisfiability problem for $AB\bar{B}\bar{L}$

In this section, we discuss the complexity of the satisfiability problem for $AB\bar{B}\bar{L}$ interpreted over strongly discrete interval temporal structures. An EXPSPACE lower bound on the complexity follows from the reduction of the *exponential-corridor tiling problem* (which is known to be EXPSPACE-complete [11]) to the satisfiability problem for the fragment $AB\bar{B}$ given in [10].

To give an upper bound to the complexity we claim that the existence of a compass structure (or compass generator) $\mathcal{G}$ that features a given formula φ can be decided by verifying suitable local (and stronger) consistency conditions over all pairs of contiguous rows, in a way similar to the EXPSPACE algorithm given in [10] for $AB\bar{B}$. In this way, to check those local conditions it is sufficient to store only *(i)* a counter y with the number of the current row, *(ii)* two guessed shadings S and S' associated with the rows y and $y+1$, and *(iii)* the characteristic functions of the shadings of y and $y+1$. Since all this information needs only an exponential amount of space, the complexity of the satisfiability problem for $AB\bar{B}\bar{L}$ is in EXPSPACE. The procedure for the infinite case is depicted in Figure 4.

For the sake of brevity, given a shading S we denote with F^{π}_{S} the unique element of S such that $\mathcal{R}eq_{B}(F^{\pi}_{S}) = \emptyset$. Note that for every row y with shading S, the type of the unit interval $[y-1, y]$ is exactly F^{π}_{S}, while the type F of all other intervals in the row must contain the formula $\langle B\rangle\top$, and thus it cannot be the case that $\mathcal{R}eq_{B}(F) = \emptyset$. Given a function $c_S : S \to \{0, ..., 8|\varphi| + 14\}$ such that $c_S(F^{\pi}_{S}) \leqslant 1$, we denote with $\overline{S}$ (*extended shading*) the pair $\langle S, c_S\rangle$. In the code we use S to denote a shading, and $\overline{S}$ to denote an extended-shading. Moreover, we introduce the following stronger version of the relation $\xrightarrow{B}$:

$$F \xmapsto{B} G \quad \text{iff} \quad \begin{cases} \mathcal{R}eq_{B}(F) = \mathcal{O}bs(G) \cup \mathcal{R}eq_{B}(G) \\ \mathcal{R}eq_{\overline{B}}(G) = \mathcal{O}bs(F) \cup \mathcal{R}eq_{\overline{B}}(F) \\ \mathcal{R}eq_{\overline{L}}(F) = \mathcal{R}eq_{\overline{L}}(G). \end{cases}$$

Finally, given two extended shadings $\overline{S} = \langle S, c_S\rangle$ and $\overline{S'} = \langle S', c_{S'}\rangle$, we say that $\overline{S'}$ is a *successor* of $\overline{S}$, and we write $\overline{S} \longmapsto \overline{S'}$, if the following conditions hold:

- for every $F \in S'$ with $\mathcal{R}eq_{B}(F) \neq \emptyset$ there exists $G \in S$ with $F \xmapsto{B} G$;
- there exists a set $R \subseteq S' \times S \times \{1, ..., 8|\varphi| + 14\}$ such that for every $(F, G, n) \in R$, $F \xmapsto{B} G$, for every $F \in S'$ we have $\sum_{(F,G,n)\in R} n = c_{S'}(F)$, and for every $G \in S$ we have $\sum_{(F,G,n)\in R} n = c_S(G)$.

The second condition ensures that all the witnesses of the lower shading S are correctly transferred in the upper shading S' according to the functions c_S and $c_{S'}$. It is easy to see that, given two rows y and $y+1$ with shadings S and S', the two extended shadings $\overline{S} = \langle S, c_y\rangle$ and $\overline{S'} = \langle S', c_{y+1}\rangle$, (where c_y and c_{y+1} are the characteristic functions of y and $y+1$, respectively) are such that $\overline{S} \longmapsto \overline{S'}$.

The main procedure basically guesses two extended shadings $\overline{S}_{past}$ and $\overline{S}_{future}$ which represent the rows y_0 and y_2 of a compass generator, and then it checks whether a compass generator featuring them exists. The procedure $\mathtt{checkPast}$ ensures that we can construct the portion of the compass structure between y_0 and y_1 (see Figure 4). The procedure starts from y_0 and construct this portion incrementally row by row until it reaches y_1. The procedure exits successfully when it reaches, without exceeding the given number of steps, a row labelled with the extended shading $\overline{S}_{past}$ and such that all formulas $\psi \in \mathcal{R}eq_{\overline{L}}(F^{\pi}_{S_{past}})$ are "witnessed" by points with the first coordinate greater than the starting row (i.e., points belonging to the red triangle in Figure 4) to guarantee that there exists a past witness set for y_1 that respects condition **G3** of Definition 10. This condition is verified

```
let ϕ be an input formula
let φ be (ϕ ∧ [B]⊥) ∨ (⟨B̄⟩ϕ) ∨ (⟨B̄⟩⟨A⟩ϕ)
let M = (2|φ| + 1)^(2^(9|φ|)) · 2^(18|φ|^2 + 9|φ|)
main
  F_0 ← any atom F with Req_B(F_0) = ∅ and φ ∈ S;
  S̄_past ← any extended shading S̄ with F_0 ∈ S;
  S̄_future ← any extended shading S̄;
  if(checkPast(S̄_past, F_0)∧
     checkFinite(S̄_past, S̄_future)∧
     checkFuture(S̄_future))
   then return true
   else return false
```

```
proc checkFinite (S̄_past, S̄_future)
  S̄ ← S̄_past;
  while i ⩽ M
    S̄′ ← any shading with S̄ ⟼ S̄′;
    if (S̄′ = S̄_future)
      then return true
    S̄ ← S̄′;
    i ← i + 1;
  return false
```

```
proc checkPast (S̄_past, F_0)
  S_lower ← {F_0}
  REQ ← Req_L̄(F_0)
  i ← 0;
  S̄ ← S̄_past;
  while i ⩽ M
    REQ ← REQ \ ⋃_{F∈S_lower} Obs(F);
    S̄′ ← any extended shading with S̄ ⟼ S̄′;
    if (S̄′ = S̄_past ∧ REQ = ∅)
      then return true
    let f : S_lower → S′ be an injective function
        s.t. f(F) ⟼B F for all F ∈ S_lower;
    S′_lower ← Img(f) ∪ {F^π_S′};
    S_lower ← S′_lower;
    S̄ ← S̄′;
    i ← i + 1;
  return false
```

```
proc checkFuture (S_future)
  for all F ∈ S_future
    REQ_F ← Req_B̄(F);
    F_above ← F;
  S̄ ← S̄_future;
  while i ⩽ M
    S̄′ ← any shading with S̄ ⟼ S̄′;
    for all F ∈ S_future
      F′_above ← any atom F ∈ S′ s.t. F ⟼B F_above;
      F_above ← F′_above;
      REQ_F ← REQ_F \ Obs(F_above);
    S̄ ← S̄′;
    if (S̄ = S̄_future ∧ ∀F ∈ S_future(REQ_F = ∅))
      then return true
    i ← i + 1;
  return false
```

Figure 5. the procedure for checking the satisfiability of ϕ over the integers.

by means of the set S_{lower} which keeps track of such points. The procedure $\mathsf{checkFinite}$ simply checks if the extended shading $\overline{S}_{future}$ is "reachable" from the extended shading $\overline{S}_{past}$, and thus it represents the construction of the "finite part" of a compass generator, that is, the portion between y_1 and y_2 in Figure 4. Finally the the procedure $\mathsf{checkFuture}$ ensures that we can construct the portion between y_2 and $N-1$ of a compass generator. This last procedure is similar to the procedure $\mathsf{checkPast}$, and it checks whether there exists a portion of a compass structure where both the lowest and the biggest rows are labelled with $\overline{S}_{future}$. To guarantee that a future witness set for y_2 exists (condition **G6** of Definition 10), we require that for every $F \in S_{future}$ and for every $\psi \in \mathcal{R}eq_{\overline{B}}(F)$, it is the case that ψ is fulfilled by some successor of S_{future}. This condition is ensured by means of the set REQ_F, which keeps track of the formulas in $\mathcal{R}eq_{\overline{B}}(F)$ that still need to be satisfied. It is worth to notice that all the counters, the extended shadings, and the shadings using in these procedures can be represented using exponential space with respect to the length of the input formula. Summing up, we obtain the following tight complexity result.

Theorem 15. *The satisfiability problem for* $A B \overline{B} \overline{L}$ *interpreted over strongly complete linear orders is EXPSPACE-complete.*

5. Embedding Metric Constraints

In the quest of finding more and more expressive, yet decidable, temporal logic for intervals, some metric extensions of decidable HS fragments have recently appeared in the literature. We refer, in particular, to [5], where the mono-modal fragment of PNL (called RPNL) has been extended with a family of metric operators that constrain the length of intervals, preserving decidability, and to [3], where the results have been extended to full PNL. Since our logic can be seen as an extension of RPNL, it makes sense to study whether it is possible to extend $\mathsf{AB\bar{B}\bar{L}}$ with metric features in the same way, preserving decidability.

A *metric constraint* is a special kind of atomic formula of the type $\mathsf{len}_{<k}$ (resp., $\mathsf{len}_{=k}$, $\mathsf{len}_{>k}$, $\mathsf{len}_{\leqslant k}$, $\mathsf{len}_{\geqslant k}$), where k is a natural number. Its semantics is very intuitive:

$$\mathcal{S}, I \models \mathsf{len}_{\sim k} \text{ iff } x - y \sim k, \text{ with } \sim \in \{=, <, >, \leqslant, \geqslant\}.$$

As noticed [5], it is sufficient to have only one type of constraint in order to express all the others; for example, $\mathsf{len}_{\geqslant k}$ is equivalent to $\neg(\mathsf{len}_{<k})$. It is easy to see that there is a simple, exponential, embedding of metric constraints in any language that includes $\langle B\rangle$. In fact, we have that $\mathsf{len}_{<k}$ is equivalent to $\neg(\underbrace{\langle B\rangle \ldots \langle B\rangle}_{k-1} \top)$. This implies that, if k is represented in binary, we have that it is possible to encode metric constraints by non-metric formulas that are exponential in the length of the metric ones. In the rest of this section, we will prove that such an exponential blow-up is not necessary, at the expenses of extending the language with enough new propositional variables.

First of all, we define the *universal modality* in the language of $\mathsf{AB\bar{B}\bar{L}}$:

$$[U]\psi \equiv \psi \wedge [A]\psi \wedge [A][A]\psi \wedge [\bar{B}]\psi \wedge [\bar{L}](\psi \wedge [\bar{B}]\psi \wedge [A]\psi \wedge [A][A]\psi)$$

Now, let $k_1 < \ldots < k_n$ be all the constants used in the set of metric constraints that we have to translate. Define $h = \lceil \log_2(k_n) \rceil$. Clearly, anyone of the constants k_i can be encoded using a string of exactly h bits; to this purpose, we use h new propositional variables $p_1, \ldots p_h$. We use such variables to label each point of our interval structure with a unique number between 0 and $2^h - 1$: to this end, we force all the intervals which share the same starting point to agree on the values of the variables $p_1, \ldots, p_n$. Moreover, we guarantee that this enumeration is ascending, that is, that intervals starting with two consecutive points are associated with two consecutive numbers, with the exception of points associated with the number $2^h - 1$, for which we set its immediate successor to be associated with 0. This is obtained by the formula

$$\psi_1 \equiv [U]((p_i \to [B]p_i) \wedge (\neg p_i \to [B]\neg p_i) \wedge ([B]\bot \to \psi_+^1))$$

where ψ_+^i is

$$\psi_+^i \equiv (p_i \to \langle A\rangle \neg p_i \wedge \psi_+^{i+1}) \wedge (\neg p_i \to \langle A\rangle p_i \bigwedge_{j=i+1}^{h} (p_j \leftrightarrow \langle A\rangle p_j)) \text{ if } 1 \leqslant i < h, \text{ and}$$

$$\psi_+^h \equiv (p_h \leftrightarrow \langle A\rangle \neg p_h) \text{ otherwise.}$$

It is worth to notice that the length of the formula ψ^1 is polynomial (more precisely, quadratic) in h, since, every formula of the type ψ_+^{i+1} is of linear length in h, and it appears exactly once in ψ^1.

In this setting, we can correctly encode the length any intervals up to $2^h - 1$, by using h new propositional variables $l_1, \ldots, l_h$, by simply looking at the numbers associated to the two endpoints of every interval. For intervals with length greater than $2^h - 1$ we can safely associate the length $2^h - 1$ (the constraint $\mathsf{len}_{\geqslant k}$ is always true on such intervals). For every interval $I = [x, y]$ of the structure, let $n(x)$ and $n(y)$ be the two numbers $0 \leqslant n(x), n(y) \leqslant 2^h - 1$ associated with the endpoints of I; the following cases may arise:

- $n(y) > n(x)$ and for every $x < z < y$ we have $n(z) \neq 0$: then, the length of I is $n(y) - n(x)$;
- $n(y) < n(x)$ and there exists exactly one $x < z < y$ such that $n(z) \neq 0$: then the length of I is $n(y) + (2^h - n(x))$;
- otherwise, the length of I is greater than $2^h - 1$.

The length of an interval I can be computed using a combination of boolean arithmetic and $\mathsf{AB\bar{B}\bar{L}}$ operators. To simplify the encoding, we observe that the bitwise complement of $n(x)$ corresponds to the quantity $(2^h - 1) - n(x)$, and thus that to compute the quantity $2^h - n(x)$ it is sufficient to add 1 to the complement of $n(x)$. Moreover, h bits are sufficient to store the result, since by hypothesis $n(x) > 1$.

For the sake of readability we introduce $2 \cdot h$ additional new variables $r_1, \ldots, r_h, c_1, \ldots, c_h$ to represent the remainders of the bit-wise operations (r variables) and the representations of the complement plus 1 (c variables). These conditions are captured by the formula

$$\psi_2 = [U]((\psi_> \rightarrow \psi_{diff}) \wedge (\psi_< \rightarrow \psi_{sum}) \wedge (\neg\psi_< \wedge \neg\psi_> \rightarrow \bigwedge_{i=1}^{h} l_i)),$$

where:

$$\begin{gathered}
\psi_< \equiv [B]\langle A\rangle\neg\psi_{zero} \wedge \psi_<^h \\
\psi_{zero} \equiv \bigwedge_{j=1^h} \neg p_j \\
\psi_<^i \equiv (\langle A\rangle\neg p_i \rightarrow \neg p_i) \wedge ((p_i \leftrightarrow \langle A\rangle p_i) \rightarrow \psi_<^{i-1}), \text{ for } 1 < i \leqslant h \\
\psi_<^1 \equiv \langle A\rangle p_i \wedge \neg p_i \\
\psi_> \equiv (\langle A\rangle\psi_{zero} \vee (\langle B\rangle\langle A\rangle\psi_{zero} \wedge [B](\langle A\rangle\psi_{zero} \rightarrow [B]\langle A\rangle\neg\psi_{zero})) \wedge \psi_>^h \\
\psi_>^i \equiv (\neg p_i \rightarrow \langle A\rangle\neg p_i) \wedge ((p_i \leftrightarrow \langle A\rangle p_i) \rightarrow \psi_<^{i-1}), \text{ for } 1 < i \leqslant h \\
\psi_<^1 \equiv \langle A\rangle p_i \wedge \neg p_i
\end{gathered}$$

$$\psi_{diff} \equiv (l_1 \leftrightarrow p_1 \oplus \langle A \rangle p_1) \wedge (r_1 \leftrightarrow p_1 \wedge \langle A \rangle \neg p_1) \wedge \psi^2_{diff}$$

$$\psi^i_{diff} \equiv \begin{array}{c} (l_i \leftrightarrow \langle A \rangle p_i \oplus (p_i \oplus r_{i-1}))\wedge \\ (r_i \leftrightarrow (r_{i-1} \wedge p_i) \vee (\neg \langle A \rangle p_i \wedge (\neg r_{i-1} \oplus \neg p_i))) \wedge \psi^{i+1}_{diff}, \text{ for } 1 < i \leqslant h \end{array}$$

$$\psi_{sum} \equiv \psi_{comp} \wedge (l_1 \leftrightarrow c_1 \oplus \langle A \rangle p_1) \wedge (r_1 \leftrightarrow c_1 \wedge \langle A \rangle p_1) \wedge \psi^2_{sum}$$

$$\psi^i_{sum} \equiv \begin{array}{c} (l_i \leftrightarrow r_{i-1} \oplus (c_i \oplus \langle A \rangle p_i))\wedge \\ (r_i \leftrightarrow (c_i \wedge \langle A \rangle p_i) \vee (r_{i-1} \wedge (c_i \vee \langle A \rangle p_i))) \wedge \psi^{i+1}_{sum}, \text{ for } 1 < i \leqslant h \end{array}$$

$$\psi_{comp} \equiv (\neg p_1 \rightarrow c_1 \bigwedge_{j=2}^{h} (c_j \leftrightarrow \neg p_j) \wedge (p_1 \rightarrow c_1 \wedge \psi^2_{comp})$$

$$\psi^i_{comp} \equiv (\neg p_i \rightarrow c_i \bigwedge_{j=i+1}^{h} (c_j \leftrightarrow \neg p_j) \wedge (p_i \rightarrow c_i \wedge \psi^{i+1}_{comp}), \text{ for } 1 < i \leqslant h.$$

Finally, for every $k_1 < \ldots < k_n$, we consider the corresponding metric constraint $\mathsf{len}_{<k_j}$ as a pure propositional variable, and set the correct interpretation by means of the following formula:

$$\psi_3 = [U](\bigwedge_{j=1}^{n} \mathsf{len}_{<k_j} \leftrightarrow (\psi^{\neq}_{k_j} \wedge [B]\psi^{\neq}_{k_j})),$$

where $\psi^{\neq}_{k_j}$ forces the length of the current interval to be different from k_j:

$$\psi^{\neq}_{k_j} \equiv \neg \bigwedge_{i=1}^{h} l_i \leftrightarrow \mathtt{bit}_i(k_j).$$

It is now easy to see that every metric constraint of the type $\mathsf{len}_{<k_j}$ holds only over intervals of length strictly less than k_i. Moreover, since the length of ψ_1, ψ_2 and ψ_3 is polynomial in h, we have proved that any formula of $AB\bar{B}\bar{L}$ with metric constraints can be translated to a non-metric one with only a polynomial blowup in the size of the formula. As a direct consequence of this, we have that adding metric constraints to $AB\bar{B}\bar{L}$ preserves decidability, without increasing the complexity class of the satisfiability problem for the language.

6. Conclusions

We considered an interval temporal logic ($AB\bar{B}\bar{L}$) with four modalities, corresponding, respectively, to Allen's interval relations *meets*, *begins*, *begun-by*, and *before*, and interpreted in the class of all strongly discrete linearly ordered sets, which includes, among others, all frames built over $\mathbb{N}$, $\mathbb{Z}$, and all finite orders. We showed that this logic is decidable in EXPSPACE, and complete for this class. The importance of this result relies on the fact that, for the considered interpretations, this logic is maximal with respect to decidability. We also showed that metric constraints of metric languages such as RPNL+INT can be polynomially embedded into this language. These results represent a non-trivial contribution towards the complete classification of all fragments of Halpern and Shoham's modal logic of intervals. We plan to complete the study of this particular language when it is interpreted over other classes of orders, such as the class of all dense linearly ordered sets, or the class of all linear orders, and to refine these results to include point-intervals, too.

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