Begin, After, and Later: a Maximal Decidable Interval Temporal Logic*

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Interval temporal logics (ITLs) are logics for reasoning about temporal statements expressed over intervals, i.e., periods of time. The most famous ITL studied so far is Halpern and Shoham's HS, which is the logic of the thirteen Allen's interval relations. Unfortunately, HS and most of its fragments have an undecidable satisfiability problem. This discouraged the research in this area until recently, when a number non-trivial decidable ITLs have been discovered. This paper is a contribution towards the complete classification of all different fragments of HS. We consider different combinations of the interval relations *begins* (B), *after* (A), *later* (L) and their inverses $\overline{\mathsf{A}}$, $\overline{\mathsf{B}}$ and $\overline{\mathsf{L}}$. We know from previous works that the combination $\mathsf{AB}\overline{\mathsf{B}}\overline{\mathsf{A}}$ is decidable only when finite domains are considered (and undecidable elsewhere), and that $\mathsf{AB}\overline{\mathsf{B}}$ is decidable over the natural numbers. We extend these results by showing that decidability of $\mathsf{AB}\overline{\mathsf{B}}$ can be further extended to capture the language $\mathsf{AB}\overline{\mathsf{B}}\overline{\mathsf{L}}$, which lays in between $\mathsf{AB}\overline{\mathsf{B}}$ and $\mathsf{AB}\overline{\mathsf{B}}\overline{\mathsf{A}}$, and that turns out to be maximal w.r.t decidability over strongly discrete linear orders (e.g. finite orders, the naturals, the integers). We also prove that the proposed decision procedure is optimal with respect to the complexity class.

1 Introduction

Interval temporal logics (ITLs) are logics for reasoning about temporal statements expressed over intervals instead of points. The most famous ITL studied so far is probably Halpern and Shoham's HS [6], which is the logic of (the thirteen) Allen's interval relations between intervals [1]. It features a modal operator for each relation, that is *meets* ($\langle\mathsf{A}\rangle$), *begins* ($\langle\mathsf{B}\rangle$), *finishes* ($\langle\mathsf{E}\rangle$), *overlaps* ($\langle\mathsf{O}\rangle$), *during* ($\langle\mathsf{D}\rangle$), *later* ($\langle\mathsf{L}\rangle$), and their inverses (denoted by $\langle\overline{\mathsf{X}}\rangle$, where $\langle\mathsf{X}\rangle$ is a modal operator), although some of them are definable in terms of others. Since HS is undecidable when interpreted over almost all interesting classes of linearly ordered sets, it is natural to ask whether there exist decidable fragments of it, and how the properties of the underlying linearly ordered domain can influence its decidable/undecidable status. In the literature, the classes of linear orderings that have received more attention are i) the class of all linearly ordered sets, ii) the set of all discrete linearly ordered sets, iii) the class of all dense linearly ordered sets. In the second case one can also distinguish among *strong* discreteness (i.e., $\mathbb{N}, \mathbb{Z}$-like), and *weak* discreteness (which allows non-standard models such as $\mathbb{N}+\mathbb{N}$).

In recent years, a number of papers have been published in which new, sometimes unexpected, decidable and undecidable fragments are presented. Among them, we mention the fragment $\mathsf{A}\overline{\mathsf{A}}$, also known as PNL, presented in [4], and studied also in [3], which is decidable over all interesting classes of models; and the fragment $\mathsf{AB}\overline{\mathsf{B}}$ (and, by symmetry, $\overline{\mathsf{A}}\mathsf{E}\overline{\mathsf{E}}$) which is decidable when interpreted over natural numbers [9]. Interestingly enough, the extension $\mathsf{AB}\overline{\mathsf{B}}\overline{\mathsf{A}}$ (and $\mathsf{AE}\overline{\mathsf{E}}\overline{\mathsf{A}}$) turns out to be decidable only

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when finite models are considered, and undecidable as soon as an infinite ascending (resp., descending) chain is admitted in the model [7]. Other interesting fragments are $\mathsf{B}\bar{\mathsf{B}}$ and $\mathsf{E}\bar{\mathsf{E}}$, that are decidable in most cases [5], while any other combination of the four operators B, $\bar{\mathsf{B}}$, E, and $\bar{\mathsf{E}}$ immediately leads to undecidability [2]. Other combinations such as $\bar{\mathsf{A}}\mathsf{B}\bar{\mathsf{B}}$, and the simpler $\bar{\mathsf{A}}\mathsf{B}$, though, remain still uncovered.

In this paper, we present another piece of this complicate puzzle by considering also the Allen's relation *later*, that captures any interval starting at some point after the ending point of the current interval, and it can be defined as $\langle \mathsf{A} \rangle \langle \mathsf{A} \rangle$, and the inverse relation *before*. We will show that the logic $\mathsf{AB}\bar{\mathsf{B}}\bar{\mathsf{L}}$ (and the symmetric logic $\bar{\mathsf{A}}\mathsf{E}\bar{\mathsf{E}}\mathsf{L}$) is decidable and EXPSPACE-complete when interpreted over strongly discrete linear orders. It is worth emphasizing that adding any other non-definable Allen's relation to $\mathsf{AB}\bar{\mathsf{B}}\bar{\mathsf{L}}$ and to $\bar{\mathsf{A}}\mathsf{E}\bar{\mathsf{E}}\mathsf{L}$ leads to undecidability over all considered structures, expect for A and $\bar{\mathsf{A}}$ that maintain decidability only when finite models are considered (and cause undecidability over infinite models). Hence, our results shows also that $\mathsf{AB}\bar{\mathsf{B}}\bar{\mathsf{L}}$ and $\bar{\mathsf{A}}\mathsf{E}\bar{\mathsf{E}}\mathsf{L}$ are maximal fragments of HS with respect to decidability in the class of all strongly discrete linear orders.

The structure of this paper is as follows. In Section 2 we introduce syntax and semantics of our logic. In Section 3, we deal with the decidability of the satisfiability problem over finite and infinite structures, while in Section 4 we discuss its complexity. Finally, in Section 5 we draw some conclusions and outline future research directions.

2 The interval temporal logic $\mathsf{AB}\bar{\mathsf{B}}\bar{\mathsf{L}}$

In this section, we briefly introduce syntax and semantics of the logic $\mathsf{AB}\bar{\mathsf{B}}\bar{\mathsf{L}}$ and we introduce the basic notions of atom, type, and dependency. We conclude the section by providing an alternative interpretation of $\mathsf{AB}\bar{\mathsf{B}}\bar{\mathsf{L}}$ over labeled grid-like structures.

2.1 Syntax and semantics

The logic $\mathsf{AB}\bar{\mathsf{B}}\bar{\mathsf{L}}$ features four modal operators $\langle \mathsf{A} \rangle$, $\langle \mathsf{B} \rangle$, $\langle \bar{\mathsf{B}} \rangle$ and $\langle \bar{\mathsf{L}} \rangle$, and it is interpreted in interval temporal structures over a strongly discrete linear order endowed with the four Allen's relations A ("meets"), B ("begins"), $\bar{\mathsf{B}}$ ("begun by") and $\bar{\mathsf{L}}$ ("before"). We recall that a linear order $\mathbb{O} = \langle O, < \rangle$ is *strongly discrete* if and only if there are only finitely many points between any pair of points $x < y \in O$. Example of strongly discrete linear orders are all finite linear orders, and the sets $\mathbb{N}$ and $\mathbb{Z}$.

Given a set $\mathcal{P}rop$ of propositional variables, formulas of $\mathsf{AB}\bar{\mathsf{B}}\bar{\mathsf{L}}$ are built up from $\mathcal{P}rop$ using the boolean connectives $\neg$ and $\vee$ and the unary modal operators $\langle \mathsf{A} \rangle$, $\langle \mathsf{B} \rangle$, $\langle \bar{\mathsf{B}} \rangle$, $\langle \bar{\mathsf{L}} \rangle$. As usual, we shall take advantage of shorthands like $\varphi_1 \wedge \varphi_2 = \neg(\neg\varphi_1 \vee \neg\varphi_2)$, $[\mathsf{A}]\varphi = \neg\langle \mathsf{A} \rangle \neg\varphi$, $[\mathsf{B}]\varphi = \neg\langle \mathsf{B} \rangle \neg\varphi$, etc. Hereafter, we denote by $|\varphi|$ the size of φ. Given any strongly discrete linear order $\mathbb{O} = \langle O, < \rangle$ we define $\mathbb{I}_{\mathbb{O}}$ as the set of all closed intervals $[x, y]$, with $x, y \in O$ and $x < y$. For any pair of intervals $[x, y], [x', y'] \in \mathbb{I}_{\mathbb{O}}$, the Allen's relations "meets" A, "begins" B, "begun by" $\bar{\mathsf{B}}$, and "before" $\bar{\mathsf{L}}$ are defined as follows:

- **"meets" relation:** $[x, y]\ \mathsf{A}\ [x', y']$ iff $y = x'$;
- **"begins" relation:** $[x, y]\ \mathsf{B}\ [x', y']$ iff $x = x'$ and $y' < y$;
- **"begun by" relation:** $[x, y]\ \bar{\mathsf{B}}\ [x', y']$ iff $x = x'$ and $y < y'$;
- **"before" relation:** $[x, y]\ \bar{\mathsf{L}}\ [x', y']$ iff $y' < x$.

Given an *interval structure* $\mathcal{S} = (\mathbb{I}_{\mathbb{O}}, \mathsf{A}, \mathsf{B}, \bar{\mathsf{B}}, \bar{\mathsf{L}}, \sigma)$, where $\sigma : \mathbb{I}_{\mathbb{O}} \to \mathscr{P}(\mathcal{P}rop)$ is a labeling function that maps intervals in $\mathbb{I}_{\mathbb{O}}$ to sets of propositional variables, and an initial interval $I = [x, y]$, we define the semantics of an $\mathsf{AB}\bar{\mathsf{B}}\bar{\mathsf{L}}$ formula as follows:

- $\mathcal{S}, I \vDash a$ iff $a \in \sigma(I)$, for any $a \in \mathcal{P}rop$;
- $\mathcal{S}, I \vDash \neg\varphi$ iff $\mathcal{S}, I \nvDash \varphi$;
- $\mathcal{S}, I \vDash \varphi_1 \vee \varphi_2$ iff $\mathcal{S}, I \vDash \varphi_1$ or $\mathcal{S}, I \vDash \varphi_2$;
- for every relation $R \in \{A, B, \overline{B}, \overline{L}\}$, $\mathcal{S}, I \vDash \langle R \rangle \varphi$ iff there is an interval $J \in \mathbb{I}_{\mathbb{O}}$ such that $I\ R\ J$ and $\mathcal{S}, J \vDash \varphi$.

Given an interval structure $\mathcal{S}$ and a formula φ, we say that $\mathcal{S}$ *satisfies* φ (and hence φ is *satisfiable*) if there is an interval I in $\mathcal{S}$ such that $\mathcal{S}, I \vDash \varphi$. Accordingly, we define the *satisfiability problem* for $AB\overline{B}\overline{L}$ as the problem of establishing whether a given $AB\overline{B}\overline{L}$-formula φ is satisfiable.

As we have recalled in the Introduction, we have that $\mathcal{S}, I \vDash \langle \overline{L} \rangle \varphi$ iff $\mathcal{S}, I \vDash \langle \overline{A} \rangle \langle \overline{A} \rangle \varphi$, and thus that $\langle \overline{L} \rangle$ is definable in the language of $AB\overline{B}\overline{A}$. As a direct consequence of the decidability and complexity results proved in this paper, we have that the converse it is not true. Moreover, it is easy to see that the operator $\langle \overline{L} \rangle$ cannot be defined in the language of $AB\overline{B}$: the modal operators $\langle A \rangle$, $\langle B \rangle$ and $\langle \overline{B} \rangle$ allow the language to see only intervals whose endpoints are greater or equals to the endpoints of the interval were a formula is interpreted. Hence, the logic $AB\overline{B}\overline{L}$ is strictly more expressive than $AB\overline{B}$ and strictly less expressive than $AB\overline{B}\overline{A}$.

2.2 Atoms, types, and dependencies

Let $\mathcal{S} = (\mathbb{I}_{\mathbb{O}}, A, B, \overline{B}, \overline{L}, \sigma)$ be an interval structure and φ be a formula of $AB\overline{B}\overline{L}$. In the sequel, we relate intervals in $\mathcal{S}$ with respect to the set of sub-formulas of φ they satisfy. To do that, we introduce the key notions of φ-*atom* and φ-*type*.

First of all, we define the *closure* $\mathcal{C}l(\varphi)$ of φ as the set of all sub-formulas of φ and of their negations (we identify $\neg\neg\alpha$ with α, $\neg\langle A \rangle \alpha$ with $[A]\neg\alpha$, etc.). For technical reasons, we also introduce the *extended closure* $\mathcal{C}l^+(\varphi)$, which is defined as the set of all formulas in $\mathcal{C}l(\varphi)$ plus all formulas of the forms $\langle R \rangle \alpha$ and $\neg\langle R \rangle \alpha$, with $R \in \{A, B, \overline{B}, \overline{L}\}$ and $\alpha \in \mathcal{C}l(\varphi)$. A φ-*atom* is any non-empty set $\mathsf{F} \subseteq \mathcal{C}l^+(\varphi)$ such that (i) for every $\alpha \in \mathcal{C}l^+(\varphi)$, we have $\alpha \in \mathsf{F}$ iff $\neg\alpha \notin \mathsf{F}$ and (ii) for every $\gamma = \alpha \vee \beta \in \mathcal{C}l^+(\varphi)$, we have $\gamma \in \mathsf{F}$ iff $\alpha \in \mathsf{F}$ or $\beta \in \mathsf{F}$ (intuitively, a φ-*atom* is a maximal *locally consistent* set of formulas chosen from $\mathcal{C}l^+(\varphi)$). Note that the cardinalities of both sets $\mathcal{C}l(\varphi)$ and $\mathcal{C}l^+(\varphi)$ are *linear* in the number $|\varphi|$ of sub-formulas of φ, while the number of φ-atoms is *at most exponential* in $|\varphi|$ (precisely, we have $|\mathcal{C}l(\varphi)| = 2|\varphi|$, $|\mathcal{C}l^+(\varphi)| = 18|\varphi|$, and there are at most $2^{8|\varphi|}$ distinct atoms). We define $\mathcal{A}_\varphi$ as the set of all possible atoms that can be built over $\mathcal{C}l^+(\varphi)$.

We also associate with each interval $I \in \mathcal{S}$ the set of all formulas $\alpha \in \mathcal{C}l^+(\varphi)$ such that $\mathcal{S}, I \vDash \alpha$. Such a set is called φ-*type* of I and it is denoted by $\mathcal{T}ype_{\mathcal{S}}(I)$. We have that every φ-type is a φ-atom, but not vice versa. Hereafter, we shall omit the argument φ, thus calling a φ-atom (resp., a φ-type) simply an atom (resp., a type). Given an atom F, we denote by $\mathcal{O}bs(\mathsf{F})$ the set of all *observables* of F, namely, the formulas $\alpha \in \mathcal{C}l(\varphi)$ such that $\alpha \in \mathsf{F}$. Similarly, given an atom F and a relation $R \in \{A, B, \overline{B}, \overline{L}\}$, we denote by $\mathcal{R}eq_R(\mathsf{F})$ the set of all R-*requests* of F, namely, the formulas $\alpha \in \mathcal{C}l(\varphi)$ such that $\langle R \rangle \alpha \in \mathsf{F}$. Taking advantage of the above sets, we can define the following three relations between two atoms F and G:

$$\mathsf{F} \xrightarrow{A} \mathsf{G} \quad \text{iff} \quad \mathcal{R}eq_A(\mathsf{F}) = \mathcal{O}bs(\mathsf{G}) \cup \mathcal{R}eq_B(\mathsf{G}) \cup \mathcal{R}eq_{\overline{B}}(\mathsf{G})$$

$$\mathsf{F} \xrightarrow{B} \mathsf{G} \quad \text{iff} \quad \begin{cases} \mathcal{O}bs(\mathsf{F}) \cup \mathcal{R}eq_{\overline{B}}(\mathsf{F}) \subseteq \mathcal{R}eq_{\overline{B}}(\mathsf{G}) \subseteq \mathcal{O}bs(\mathsf{F}) \cup \mathcal{R}eq_{\overline{B}}(\mathsf{F}) \cup \mathcal{R}eq_B(\mathsf{F}) \\ \mathcal{O}bs(\mathsf{G}) \cup \mathcal{R}eq_B(\mathsf{G}) \subseteq \mathcal{R}eq_B(\mathsf{F}) \subseteq \mathcal{O}bs(\mathsf{G}) \cup \mathcal{R}eq_B(\mathsf{G}) \cup \mathcal{R}eq_{\overline{B}}(\mathsf{G}) \\ \mathcal{R}eq_{\overline{L}}(\mathsf{F}) = \mathcal{R}eq_{\overline{L}}(\mathsf{G}). \end{cases}$$

$$\mathsf{F} \xrightarrow{\overline{L}} \mathsf{G} \quad \text{iff} \quad \mathcal{O}bs(\mathsf{G}) \cup \mathcal{R}eq_{\overline{L}}(\mathsf{G}) \subseteq \mathcal{R}eq_{\overline{L}}(\mathsf{F})$$

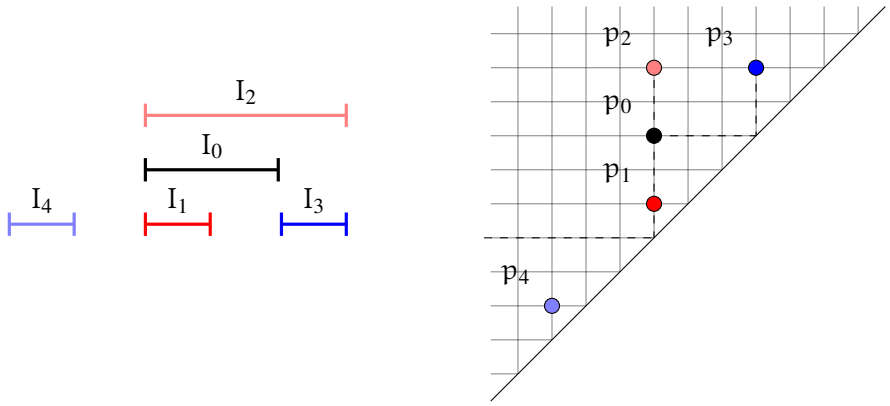

Figure 1: Correspondence between intervals and the points of a grid.

Note that the relations $\xrightarrow{B}$ and $\xrightarrow{\bar{L}}$ are transitive, while $\xrightarrow{A}$ is not. Moreover, all $\xrightarrow{A}$, $\xrightarrow{B}$ and $\xrightarrow{\bar{L}}$ satisfy a *view-to-type dependency*, namely, for every pair of intervals I, J in $\mathbb{S}$, we have that

$$\begin{array}{lll} I\ A\ J & \text{implies} & \mathcal{T}ype_{\mathbb{S}}(I) \xrightarrow{A} \mathcal{T}ype_{\mathbb{S}}(J) \\ I\ B\ J & \text{implies} & \mathcal{T}ype_{\mathbb{S}}(I) \xrightarrow{B} \mathcal{T}ype_{\mathbb{S}}(J) \\ I\ \bar{L}\ J & \text{implies} & \mathcal{T}ype_{\mathbb{S}}(I) \xrightarrow{\bar{L}} \mathcal{T}ype_{\mathbb{S}}(J). \end{array}$$

2.3 Compass structures

The logic $\mathrm{AB\bar{B}\bar{L}}$ can be equivalently interpreted over grid-like structures (hereafter called *compass structures*) by exploiting the existence of a natural bijection between the intervals $I = [x, y]$ and the points $p = (x, y)$ of an $\mathbb{O} \times \mathbb{O}$ grid such that $x < y$. As an example, in Fig. 1 are shown five intervals $I_0, ..., I_4$, such that $I_0\ B\ I_1$, $I_0\ \bar{B}\ I_2$, $I_0\ A\ I_3$, and $I_0\ \bar{L}\ I_4$, together with the corresponding points $p_0, ..., p_4$ of a grid (note that the four Allen's relations $A, B, \bar{B}, \bar{L}$ between intervals are mapped to the corresponding spatial relations between points; for the sake of readability, we name the latter ones as the former ones).

Definition 1. *Given an* $\mathrm{AB\bar{B}\bar{L}}$ *formula* φ*, a (consistent and fulfilling)* compass *(*φ*-)*structure *is a pair* $\mathcal{G} = (\mathbb{P}_{\mathbb{O}}, \mathcal{L})$*, where* $\mathbb{P}_{\mathbb{O}}$ *is the set of points of the form* $p = (x, y)$*, with* $x, y \in \mathbb{O}$ *and* $x \leqslant y$*, and* $\mathcal{L}$ *is function that maps any point* $p \in \mathbb{P}_{\mathbb{O}}$ *to a (*φ*-)atom* $\mathcal{L}(p)$ *in such a way that:*

- *for every pair of points* $p, q \in \mathbb{P}_{\mathbb{O}}$ *and every relation* $R \in \{A, B, \bar{L}\}$*, if* $p\ R\ q$ *holds, then* $\mathcal{L}(p) \xrightarrow{R} \mathcal{L}(q)$ *follows (***consistency***);*
- *for every point* $p \in \mathbb{P}_{\mathbb{O}}$*, every relation* $R \in \{A, B, \bar{B}, \bar{L}\}$*, and every formula* $\alpha \in \mathcal{R}eq_R\big(\mathcal{L}(p)\big)$*, there is a point* $q \in \mathbb{P}_{\mathbb{O}}$ *such that* $p\ R\ q$ *and* $\alpha \in \mathcal{O}bs\big(\mathcal{L}(q)\big)$ *(***fulfillment***).*

We say that a compass (φ-)structure $\mathcal{G} = (\mathbb{P}_{\mathbb{O}}, \mathcal{L})$ *features* a formula α if there is a point $p \in \mathbb{P}_{\mathbb{O}}$ such that $\alpha \in \mathcal{L}(p)$. The following proposition implies that the satisfiability problem for $\mathrm{AB\bar{B}\bar{L}}$ is reducible to the problem of deciding, for any given formula φ, whether there exists a φ-compass structure featuring φ.

Proposition 2. *An* $\mathrm{AB\bar{B}\bar{L}}$*-formula* φ *is satisfied by some interval structure if and only if it is featured by some (*φ*-)compass structure.*

3 Deciding the satisfiability problem for $\mathrm{AB}\overline{\mathrm{B}}\overline{\mathrm{L}}$

In this section, we prove that the satisfiability problem for $\mathrm{AB}\overline{\mathrm{B}}\overline{\mathrm{L}}$ is decidable by providing a "small-model theorem" for the satisfiable formulas of the logic. For the sake of simplicity, we first show that the satisfiability problem for $\mathrm{AB}\overline{\mathrm{B}}\overline{\mathrm{L}}$ interpreted over *finite* interval structures is decidable and then we generalize such a result to all (finite or infinite) interval structures based on strong discrete linear orders.

As a preliminary step, we introduce the key notions of shading, of witness set, and of compatibility between rows of a compass structure. Let $\mathcal{G} = (\mathbb{P}_{\mathbb{O}}, \mathcal{L})$ be a compass structure and let $y \in \mathbb{O}$. The *shading of the row* y *of* $\mathcal{G}$ is the set $\mathcal{S}\mathit{hading}_{\mathcal{G}}(y) = \{\mathcal{L}(x,y) : x < y\}$, namely, the set of the atoms of all points in $\mathbb{P}_{\mathbb{O}}$ whose vertical coordinate has value y (basically, we interpret different atoms as different colors). A *witness set for* y is any *minimal* set $\mathcal{W}\mathit{it}(y) \subseteq \{(x_\psi, y_\psi) : x_\psi < y_\psi \wedge y_\psi > y\}$ that respects the following property:

(WIT) for every $\psi \in \mathcal{C}l(\varphi)$ that appears in the labeling of some point (x', y') with $y' > y$, there exists a *witness* $(x_\psi, y_\psi) \in \mathcal{W}\mathit{it}(y)$ such that

1. $\psi \in \mathcal{L}(x_\psi, y_\psi)$, and
2. y_ψ is minimal, that is, for all (x', y') with $y < y' < y_\psi$, $\neg\psi \in \mathcal{L}(x', y')$.

Since $\mathcal{W}\mathit{it}(y)$ is minimal we have that there is at most one distinct point for every $\psi \in \mathcal{C}l(\varphi)$ and thus $|\mathcal{W}\mathit{it}(y)| < |\mathcal{C}l(\varphi)| = 2 \cdot |\varphi|$. Intuitively, a witness set for a row y is a set that contains, for every formula ψ that occurs in some point above the row y, a witness (x_ψ, y) for it, that is, a point that satisfies ψ at the minimum possible distance from the row y. The notion of shading and of witness set allow us to determine whether two rows are *compatible* or not.

Let $P \subseteq \mathbb{P}_{\mathbb{O}}$ a set of points and $\bar{y}$ be a coordinate we define $\pi_{\bar{y}}(P) = \{x : (x, y) \in P \wedge y < \bar{y}\}$, the set of all x-coordinate belonging to points in P which are smaller than y.

DEFINITION 3. *Given a compass structure* $\mathcal{G}$ *and two rows* $y_0 < y_1$*, we say that* y_0 *and* y_1 *are* compatible *if and only if the following properties holds:*

1. $\mathcal{S}\mathit{hading}_{\mathcal{G}}(y_0) = \mathcal{S}\mathit{hading}_{\mathcal{G}}(y_1)$;
2. $\mathcal{L}(y_0 - 1, y_0) = \mathcal{L}(y_1 - 1, y_1)$;
3. *there exists a witness set* $\mathcal{W}\mathit{it}(y_1)$ *for* y_1 *and an* injective mapping function $w : \pi_{y_1}(\mathcal{W}\mathit{it}(y_1)) \mapsto \{x : x < y_0\}$ *s.t.* $\mathcal{L}(x, y_1) = \mathcal{L}(w(x), y_0)$ *for every* $x \in \pi_{y_1}(\mathcal{W}\mathit{it}(y_1))$*, that assigns a distinct* x*-coordinate on the row* y_0 *for every witness* (x_ψ, y_ψ) *in* $\mathcal{W}\mathit{it}(y_1)$ *with* $x_\psi \leqslant y_1$.

In the following, we will show how the properties of compatible rows can be used to contract compass structures to smaller ones, first for finite models and then for infinite ones.

3.1 A small-model theorem for finite structures

Let φ be an $\mathrm{AB}\overline{\mathrm{B}}\overline{\mathrm{L}}$ formula. It is easy to see that φ is satisfiable over a finite model if and only if the formula $\varphi \vee \langle\overline{\mathrm{B}}\rangle\varphi \vee \langle\mathrm{A}\rangle\varphi \vee \langle\mathrm{A}\rangle\langle\mathrm{A}\rangle\varphi$ is featured by the *initial point* $(0,1)$ a *finite* compass structure $\mathcal{G} = (\mathbb{P}_{\mathbb{O}}, \mathcal{L})$. We prove that we can restrict our attention to compass structures $\mathcal{G} = (\mathbb{P}_{\mathbb{O}}, \mathcal{L})$ with a number of points in $\mathbb{O}$ bounded by a double exponential in $|\varphi|$. We start with the following lemma that proves two simple, but crucial, properties of the relations $\xrightarrow{\mathrm{A}}$, $\xrightarrow{\mathrm{B}}$, and $\xrightarrow{\bar{\mathrm{L}}}$.

LEMMA 4. *Let* F, G, H *be some atoms:*

1. *if* $F \xrightarrow{\mathrm{A}} H$ *and* $G \xrightarrow{\mathrm{B}} H$ *hold, then* $F \xrightarrow{\mathrm{A}} G$ *holds as well;*
2. *if* $F \xrightarrow{\mathrm{B}} G$ *and* $G \xrightarrow{\bar{\mathrm{L}}} H$ *hold, then* $F \xrightarrow{\bar{\mathrm{L}}} H$ *holds as well.*

Proof. The proof for property 1 can be found in [8]. As for property 2, we have that, by the definition of $\xrightarrow{B}$, if $F \xrightarrow{B} G$ then $\mathcal{R}eq_{\bar{L}}(F) = \mathcal{R}eq_{\bar{L}}(G)$. This implies that $\mathcal{O}bs(H) \cup \mathcal{R}eq_{\bar{L}}(H) \subseteq \mathcal{R}eq_{\bar{L}}(F)$ and thus $F \xrightarrow{\bar{L}} H$ holds as well. □

The next lemma shows that, under suitable conditions, a given compass structure $\mathcal{G}$ may be reduced in length, preserving the existence of atoms featuring φ.

LEMMA 5. *Let $\mathcal{G}$ be a finite compass structure of size N featuring φ on the initial point $(0,1)$. If there exist two compatible rows $0 < y_0 < y_1 < N$ in $\mathcal{G}$, then there exists a compass structure $\mathcal{G}'$ of size $N' = N - y_1 + y_0$ that features φ.*

Proof. Suppose that $0 < y_0 < y_1 < N$ are two compatible rows of $\mathcal{G}$. By definition, we have that $\mathcal{S}hading_{\mathcal{G}}(y_0) = \mathcal{S}hading_{\mathcal{G}}(y_1)$, $\mathcal{L}(y_0 - 1, y_0) = \mathcal{L}(y_1 - 1, y_1)$, and there exists a witness set $\mathcal{W}it(y_1)$ for y_1 and an *injective mapping function* $w : \pi_{y_1}(\mathcal{W}it(y_1)) \mapsto \{x : x < y_0\}$ that assigns a distinct x-coordinate on the row y_0 for every witness (x_ψ, y_ψ) in $\mathcal{W}it(y_1)$ with $x_\psi \leqslant y_1$. Then, we can define a function $f : \{0, ..., y_0 - 1\} \mapsto \{0, ..., y_1 - 1\}$ such that, for every $0 \leqslant x < y_0$, $\mathcal{L}(x, y_0) = \mathcal{L}(f(x), y_1)$ and for every $(x_\psi, y_\psi) \in \mathcal{W}it(y_1)$ if $x_\psi < y_1$ then $f(w(x_\psi)) = x_\psi$.

Let $k = y_1 - y_0$, $N' = N - k$ ($< N$), $\mathbb{O}' = \langle \{0, \ldots, N' - 1\}, < \rangle$, and $\mathbb{P}_{\mathbb{O}'}$ be the correspondent portion of the grid. We extend f to a function that maps points in $\mathbb{P}_{\mathbb{O}'}$ to points in $\mathbb{P}_{\mathbb{O}}$ as follows:

- if $p = (x, y)$, with $0 \leqslant x < y < y_0$, then we simply let $f(p) = p$;
- if $p = (x, y)$, with $0 \leqslant x < y_0 \leqslant y$, then we let $f(p) = (f(x), y + k)$;
- if $p = (x, y)$, with $y_0 \leqslant x < y$, then we let $f(p) = (x + k, y + k)$.

We denote by $\mathcal{L}'$ the labeling of $\mathbb{P}_{\mathbb{O}'}$ such that, for every point $p \in \mathbb{P}_{\mathbb{O}'}$, $\mathcal{L}'(p) = \mathcal{L}(f(p))$ and we denote by $\mathcal{G}'$ the resulting structure $(\mathbb{P}_{\mathbb{O}'}, \mathcal{L}')$ (see Figure 2). We have to prove that $\mathcal{G}'$ is a consistent and fulfilling compass structure that features φ. First, we show that $\mathcal{G}'$ satisfies the consistency conditions for the relations B, A, and $\bar{L}$; then we show that $\mathcal{G}'$ satisfies the fulfillment conditions for the $\bar{B}$-, B-, A, and $\bar{L}$-requests; finally, we show that $\mathcal{G}'$ features φ.

CONSISTENCY WITH RELATION B. Consider two points $p = (x, y)$ and $p' = (x', y')$ in $\mathcal{G}'$ such that p B p', i.e., $0 \leqslant x = x' < y' < y < N'$. We prove that $\mathcal{L}'(p) \xrightarrow{B} \mathcal{L}'(p')$ by distinguishing among the following three cases (note that exactly one of such cases holds):

1. $y < y_0$ and $y' < y_0$,
2. $y \geqslant y_0$ and $y' \geqslant y_0$,
3. $y \geqslant y_0$ and $y' < y_0$.

If $y < y_0$ and $y' < y_0$, then, by construction, we have $f(p) = p$ and $f(p') = p'$. Since $\mathcal{G}$ is a (consistent) compass structure, we immediately obtain $\mathcal{L}'(p) = \mathcal{L}(p) \xrightarrow{B} \mathcal{L}(p') = \mathcal{L}'(p')$.

If $y \geqslant y_0$ and $y \geqslant y_0$, then, by construction, we have either $f(p) = (f(x), y + k)$ or $f(p) = (x + k, y + k)$, depending on whether $x < y_0$ or $x \geqslant y_0$. Similarly, we have either $f(p') = (f(x'), y' + k) = (f(x), y' + k)$ or $f(p') = (x' + k, y' + k) = (x + k, y' + k)$. This implies $f(p)$ B $f(p')$ and thus, since $\mathcal{G}$ is a (consistent) compass structure, we have $\mathcal{L}'(p) = \mathcal{L}(f(p)) \xrightarrow{B} \mathcal{L}(f(p')) = \mathcal{L}'(p')$.

If $y \geqslant y_0$ and $y' < y_0$, then, since $x < y' < y_0$, we have by construction $f(p) = (f(x), y + k)$ and $f(p') = p'$. Moreover, if we consider the point $p'' = (x, y_0)$ in $\mathcal{G}'$, we easily see that (i) $f(p'') = (f(x), y_1)$, (ii) $f(p)$ B $f(p'')$ (whence $\mathcal{L}(f(p)) \xrightarrow{B} \mathcal{L}(f(p''))$), (iii) $\mathcal{L}(f(p'')) = \mathcal{L}(p'')$, and (iv) p'' B p' (whence $\mathcal{L}(p'') \xrightarrow{B} \mathcal{L}(p')$). It thus follows that $\mathcal{L}'(p) = \mathcal{L}(f(p)) \xrightarrow{B} \mathcal{L}(f(p'')) = \mathcal{L}(p'') \xrightarrow{B} \mathcal{L}(p') = \mathcal{L}(f(p')) = \mathcal{L}'(p')$. Finally, by exploiting the transitivity of the relation $\xrightarrow{B}$, we obtain $\mathcal{L}'(p) \xrightarrow{B} \mathcal{L}'(p')$.

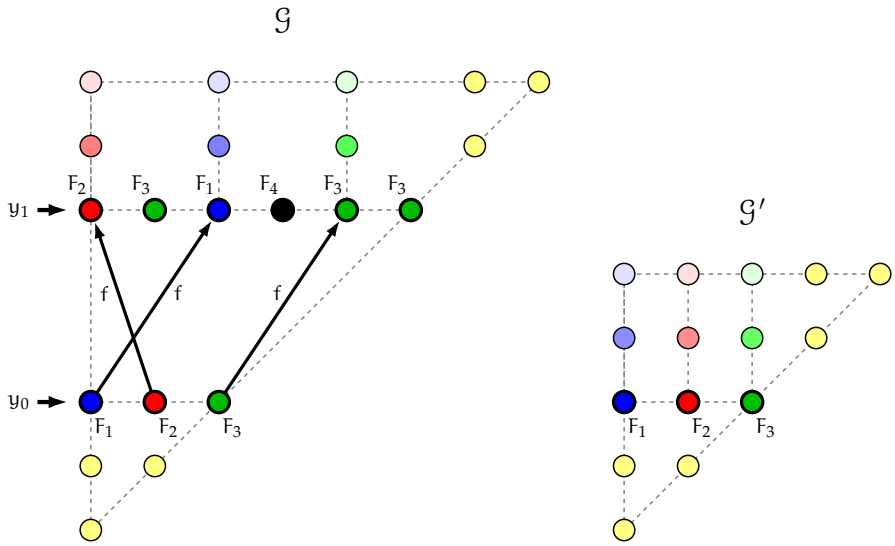

Figure 2: Contraction $\mathcal{G}'$ of a compass structure $\mathcal{G}$.

CONSISTENCY WITH RELATION A. Consider two points $p = (x,y)$ and $p' = (x',y')$ such that p A p', i.e., $0 \leqslant x < y = x' < y' < N'$. We define $p'' = (y,y+1)$ in such a way that p A p'' and p' B p'' and we distinguish between the following two cases:

1. $y \geqslant y_0$,
2. $y < y_0$.

If $y \geqslant y_0$, then, by construction, we have $f(p)$ A $f(p'')$. Since $\mathcal{G}$ is a (consistent) compass structure, it follows that $\mathcal{L}'(p) = \mathcal{L}(f(p)) \xrightarrow{A} \mathcal{L}(f(p'')) = \mathcal{L}'(p'')$.

If $y < y_0$, then, by construction, we have $\mathcal{L}(p'') = \mathcal{L}(f(p''))$. Again, since $\mathcal{G}$ is a (consistent) compass structure, it follows that $\mathcal{L}'(p) = \mathcal{L}(f(p)) = \mathcal{L}(p) \xrightarrow{A} \mathcal{L}(p'') = \mathcal{L}(f(p'')) = \mathcal{L}'(p'')$.

In both cases we have $\mathcal{L}'(p) \xrightarrow{A} \mathcal{L}'(p'')$. Now, we recall that p' B p'' and that, by previous arguments, $\mathcal{G}'$ is consistent with the relation B. We thus have $\mathcal{L}'(p') \xrightarrow{B} \mathcal{L}'(p'')$. Finally, by applying Lemma 4, we obtain $\mathcal{L}'(p) \xrightarrow{A} \mathcal{L}'(p')$.

CONSISTENCY WITH RELATION $\bar{L}$. Consider two points $p = (x,y)$ and $p' = (x',y')$ in $\mathcal{G}'$ such that p $\bar{L}$ p', i.e., $0 \leqslant x' < y' < x < y < N'$. We prove that $\mathcal{L}'(p) \xrightarrow{\bar{L}} \mathcal{L}'(p')$ by distinguishing among the following three cases (note that exactly one of such cases holds):

1. $y < y_0$ and $y' < y_0$,
2. $y \geqslant y_0$ and $y' \geqslant y_0$,
3. $y \geqslant y_0$ and $y' < y_0$.

If $y < y_0$ and $y' < y_0$, then, by construction, we have $f(p) = p$ and $f(p') = p'$. Since $\mathcal{G}$ is a (consistent) compass structure, we immediately obtain $\mathcal{L}'(p) = \mathcal{L}(p) \xrightarrow{\bar{L}} \mathcal{L}(p') = \mathcal{L}'(p')$.

If $y \geqslant y_0$ and $y' \geqslant y_0$, then, by construction, we have either $f(p') = (f(x'), y'+k)$ or $f(p') = (x'+k, y'+k)$, depending on whether $x' < y_0$ or $x' \geqslant y_0$. Since $y_0 \leqslant y' < x$, we have $f(p) = (x+k, y+k)$. This implies $f(p)$ $\bar{L}$ $f(p')$ and thus, since $\mathcal{G}$ is a (consistent) compass structure, we have $\mathcal{L}'(p) = \mathcal{L}(f(p)) \xrightarrow{\bar{L}} \mathcal{L}(f(p')) = \mathcal{L}'(p')$.

If $y \geqslant y_0$ and $y' < y_0$, then, we have by construction that $f(p') = p'$ and either $f(p) = (x+k, y+k)$ or $f(p) = (f(x), y+k)$. In the former case we have that $f(p)$ $\bar{L}$ $f(p')$ and thus, since $\mathcal{G}$ is a consistent

compass structure, $\mathcal{L}'(p) = \mathcal{L}(f(p)) \xrightarrow{\bar{L}} \mathcal{L}(f(p')) = \mathcal{L}'(p')$. In the latter case it is not necessarily true that $y' < f(x)$. Consider the points $p'' = (f(x), y_1)$ and $p''' = (x, y_0)$: by the definition of f, $\mathcal{L}(p'') = \mathcal{L}(p''')$. Moreover, we have that $f(p) B p''$ and $p''' \bar{L} f(p') = p'$. Since $\mathcal{G}$ is a consistent compass structure, this implies that $\mathcal{L}'(p) = \mathcal{L}(f(p)) \xrightarrow{B} \mathcal{L}(p'') = \mathcal{L}(p''') \xrightarrow{\bar{L}} \mathcal{L}(f(p')) = \mathcal{L}'(p')$. Finally, by applying Lemma 4, we obtain $\mathcal{L}'(p) \xrightarrow{\bar{L}} \mathcal{L}'(p')$.

FULFILLMENT OF B-REQUESTS. Consider a point $p = (x, y)$ in $\mathcal{G}'$ and some B-request $\alpha \in \mathcal{R}eq_B\big(\mathcal{L}'(p)\big)$ associated with it. Since, by construction, $\alpha \in \mathcal{R}eq_B\big(\mathcal{L}(f(p))\big)$ and $\mathcal{G}$ is a (fulfilling) compass structure, we know that $\mathcal{G}$ contains a point $q' = (x', y')$ such that $f(p)\ B\ q'$ and $\alpha \in \mathcal{O}bs\big(\mathcal{L}(q')\big)$. We prove that $\mathcal{G}'$ contains a point p' such that $p\ B\ p'$ and $\alpha \in \mathcal{O}bs\big(\mathcal{L}'(p')\big)$ by distinguishing among the following three cases (note that exactly one of such cases holds):

1. $y < y_0$
2. $y' \geqslant y_1$,
3. $y \geqslant y_0$ and $y' < y_1$.

If $y < y_0$, then, by construction, we have $p = f(p)$ and $q' = f(q')$. Therefore, we simply define $p' = q'$ in such a way that $p = f(p)\ B\ q' = p'$ and $\alpha \in \mathcal{O}bs\big(\mathcal{L}'(p')\big)$ $(= \mathcal{O}bs\big(\mathcal{L}(f(p'))\big) = \mathcal{O}bs\big(\mathcal{L}(q')\big))$.

If $y' \geqslant y_1$, then, by construction, we have either $f(p) = (f(x), y + k)$ or $f(p) = (x + k, y + k)$, depending on whether $x < y_0$ or $x \geqslant y_0$. We define $p' = (x, y' - k)$ in such a way that $p\ B\ p'$. Moreover, we observe that either $f(p') = (f(x), y')$ or $f(p') = (x + k, y')$, depending on whether $x < y_0$ or $x \geqslant y_0$, and in both cases $f(p') = q'$ follows. This shows that $\alpha \in \mathcal{O}bs\big(\mathcal{L}'(p')\big)$ $(= \mathcal{O}bs\big(\mathcal{L}(f(p'))\big) = \mathcal{O}bs\big(\mathcal{L}(q')\big))$.

If $y \geqslant y_0$ and $y' < y_1$, then we define $\bar{p} = (x, y_0)$ and $\bar{q} = (x', y_1)$ and we observe that $f(p)\ B\ \bar{q}$, $\bar{q}\ B\ q'$, and $f(\bar{p}) = \bar{q}$. From $f(p)\ B\ \bar{q}$ and $\bar{q}\ B\ q'$, it follows that $\alpha \in \mathcal{R}eq_B\big(\mathcal{L}(\bar{q})\big)$ and hence $\alpha \in \mathcal{R}eq_B\big(\mathcal{L}(\bar{p})\big)$. Since $\mathcal{G}$ is a (fulfilling) compass structure, we know that there is a point p' such that $\bar{p}\ B\ p'$ and $\alpha \in \mathcal{O}bs\big(\mathcal{L}(\bar{p}')\big)$. Moreover, since $\bar{p}\ B\ p'$, we have $f(p') = p'$, from which we obtain $p\ B\ p'$ and $\alpha \in \mathcal{O}bs\big(\mathcal{L}(p')\big)$.

FULFILLMENT OF $\bar{B}$-REQUESTS. The proof that $\mathcal{G}'$ fulfills all $\bar{B}$-requests of its atoms is symmetric with respect to the previous one.

FULFILLMENT OF A-REQUESTS. Consider a point $p = (x, y)$ in $\mathcal{G}'$ and some A-request $\alpha \in \mathcal{R}eq_A\big(\mathcal{L}'(p)\big)$ associated with p in $\mathcal{G}'$. Since, by previous arguments, $\mathcal{G}'$ fulfills all $\bar{B}$-requests of its atoms, it is sufficient to prove that either $\alpha \in \mathcal{O}bs\big(\mathcal{L}'(p')\big)$ or $\alpha \in \mathcal{R}eq_{\bar{B}}\big(\mathcal{L}'(p')\big)$, where $p' = (y, y + 1)$. This can be easily proved by distinguishing among the three cases $y < y_0 - 1$, $y = y_0 - 1$, and $y \geqslant y_0$.

FULFILLMENT OF $\bar{L}$-REQUESTS. Consider a point $p = (x, y)$ in $\mathcal{G}'$ and some $\bar{L}$-request $\alpha \in \mathcal{R}eq_{\bar{L}}\big(\mathcal{L}'(p)\big)$ associated with it. Since, by construction, $\alpha \in \mathcal{R}eq_{\bar{L}}\big(\mathcal{L}(f(p))\big)$ and $\mathcal{G}$ is a (fulfilling) compass structure, we know that $\mathcal{G}$ contains a point $q' = (x', y')$ such that $f(p)\ \bar{L}\ q'$ and $\alpha \in \mathcal{O}bs\big(\mathcal{L}(q')\big)$. To simplify the proofs, we assume that q' is *minimal* with respect to the vertical coordinate, that is, for every other point $q'' = (x'', y'')$ with $y'' < y'$, $\alpha \notin \mathcal{O}bs\big(\mathcal{L}(q'')\big)$. We prove that $\mathcal{G}'$ contains a point p' such that $p\ \bar{L}\ p'$ and $\alpha \in \mathcal{O}bs\big(\mathcal{L}'(p')\big)$ by distinguishing among the following five cases (note that exactly one of such cases holds):

1. $y \leqslant y_0$,
2. $x < y_0$ and $y \geqslant y_0$,
3. $x \geqslant y_0$ and $y' < y_1$,
4. $x \geqslant y_0$ and $y' = y_1$,
5. $x \geqslant y_0$ and $y' > y_1$.

If $y < y_0$, then, by construction, we have $p = f(p)$ and $q' = f(q')$. Therefore, we simply define $p' = q'$ in such a way that $p = f(p)\ \bar{L}\ q' = p'$ and $\alpha \in \mathcal{O}bs\big(\mathcal{L}'(p')\big)$ $(= \mathcal{O}bs\big(\mathcal{L}(f(p'))\big) = \mathcal{O}bs\big(\mathcal{L}(q')\big))$.

If $x < y_0$ and $y \geqslant y_0$ then $f(p) = (f(x), y + k)$. Now, consider the point $p'' = (f(x), y_1)$: since $f(p)Bp''$ and $\mathcal{G}$ is a consistent compass structure, we have that $\mathcal{R}eq_{\bar{L}}(p'') = \mathcal{R}eq_{\bar{L}}(f(p))$. By definition of f, we have that $\mathcal{L}(f(x), y_1) = \mathcal{L}(x, y_0)$ and thus, since $\mathcal{G}$ is fulfilling, there exists a point $p' = (x'', y'')$ such that $y'' < x$ and $\alpha \in \mathcal{O}bs\big(\mathcal{L}(p')\big)$. Since $f(p') = p'$, this shows that $\alpha \in \mathcal{O}bs\big(\mathcal{L}'(p')\big)$ as well.

If $x \geqslant y_0$ and $y' < y_1$ then $f(p) = (x + k, y + k)$. Since $\mathcal{G}$ is a consistent compass structure, we have that $\alpha \in \mathcal{R}eq_{\bar{L}}(\mathcal{L}(y_1 - 1, y_1))$. By the definition of compatible rows, we have that $\mathcal{L}(y_1 - 1, y_1) = \mathcal{L}(y_0 - 1, y_0)$ and thus (by the minimality assumption) $y' < y_0$ and $q' = f(q')$. Therefore, we simply define $p' = q'$ in such a way that $p\ \bar{L}\ q' = p'$ and $\alpha \in \mathcal{O}bs\big(\mathcal{L}'(p')\big)$ $(= \mathcal{O}bs\big(\mathcal{L}(f(p'))\big) = \mathcal{O}bs\big(\mathcal{L}(q')\big))$.

If $x \geqslant y_0$ and $y' = y_1$ then $\mathcal{L}(q') \in \mathcal{S}hading_{\mathcal{G}}(y_1)$. By the definition of compatible rows, we have that $\mathcal{S}hading_{\mathcal{G}}(y_1) = \mathcal{S}hading_{\mathcal{G}}(y_0)$ and thus there must exists a point $q'' = (x'', y_0)$ such that $\mathcal{L}(q') = \mathcal{L}(q'')$ and $y_0 < y'$, against the hypothesis that q' is a minimal point satisfying α. Hence, this case cannot happen.

If $x \geqslant y_0$ and $y' > y_1$ then, by the minimality assumption on q' we have that for every $y'' < y'$, $\alpha \notin \mathcal{O}bs\big(\mathcal{L}(x'', y'')\big)$ for any $x'' < y''$. Hence, by the definition of witness set, we have that there exists a witness $(x_\alpha, y_\alpha) \in \mathcal{W}it(y_1)$ such that $\alpha \in \mathcal{O}bs\big(\mathcal{L}(x_\alpha, y_\alpha)\big)$ and $y_\alpha = y'$ (by the minimality assumption). If $x_\alpha \geqslant y_1$ then we define $p' = (x_\alpha - k, y_\alpha - k)$. Otherwise, $x_\alpha < y_1$ and by the definition of the mapping function w and of the function f, we have that $f(w(x_\alpha)) = x_\alpha$: we define $p' = (w(x_\alpha), y' - k)$. In both cases we have that $f(p') = (x_\alpha, y_\alpha)$, $p\bar{L}p'$ and $\alpha \in \mathcal{O}bs\big(\mathcal{L}'(p')\big)$.

FEATURED FORMULAS. Recall that, by previous assumptions, $\varphi \in \mathcal{L}(0,1)$. Since our contraction procedure never changes the labelling of the initial point, $\varphi \in \mathcal{L}'(0,1)$ as well. □

On the grounds of the above result, we can provide a suitable upper bound for the length of a minimal finite interval structure that satisfies φ, if there exists any. This yields a straightforward, but inefficient, 2NEXPTIME algorithm that decides whether a given $ABB\bar{B}\bar{L}$-formula φ is satisfiable over finite interval structures.

THEOREM 6. *An $ABB\bar{B}\bar{L}$-formula φ is satisfied by some finite interval structure iff it is featured by some compass structure of length $N \leqslant (8|\varphi| + 15)^{2^{32|\varphi|+56}} \cdot 2^{32|\varphi|+56}$ (i.e., double exponential in $|\varphi|$).*

Proof. Suppose that φ is satisfied by a finite interval structure $\mathcal{S}$, and let $\xi = \varphi \vee \langle\bar{B}\rangle\varphi \vee \langle A\rangle\varphi \vee \langle A\rangle\langle A\rangle\varphi$. By Proposition 2, there is a compass structure $\mathcal{G}$ that features ξ on the initial point and has finite length N. By Lemma 5, we can assume without loss of generality that all rows of $\mathcal{G}$ are pairwise incompatible. We recall from Section 2.2 that $\mathcal{G}$ contains at most $2^{8|\xi|}$ distinct atoms. For every row y of the compass structure and every atom $F \in \mathcal{A}_\xi$, let $\#(F, y)$ be the cardinality of the set $\{(x, y) : x < y \text{ and } \mathcal{L}(x, y) = F\}$. We associate to every row y of the structure a *characteristic function* $c_y : \mathcal{A}_\xi \mapsto \mathbb{N}$ defined as follows:

$$c_y(F) \quad = \quad \begin{cases} \#(F, y) & \#(F, y) \leqslant 2|\xi| \\ 2|\xi| & \text{otherwise} \end{cases} \tag{1}$$

Since any witness set $\mathcal{W}it(y)$ contains at most $2|\xi|$ witnesses, it is easy to see that two rows y_0 and y_1 with the same characteristic function and such that $\mathcal{L}(y_0 - 1, y_0) = \mathcal{L}(y_1 - 1, y_1)$ are compatible. The number of possible characteristic functions is bounded by $(2|\xi| + 1)^{2^{8|\xi|}}$, and thus $\mathcal{G}$ cannot have more than $(2|\xi| + 1)^{2^{8|\xi|}} \cdot 2^{8|\xi|}$ rows. Since $|\xi| = 4|\varphi| + 7$ we can conclude that $N \leqslant (8|\varphi| + 15)^{2^{32|\varphi|+56}} \cdot 2^{32|\varphi|+56}$, and thus double exponential in $|\varphi|$. □

3.2 A small-model theorem for infinite structures

In general, compass structures that feature φ may be infinite. Here, we prove that, without loss of generality, we can restrict our attention to sufficiently "regular" infinite compass structures, which can be represented in double exponential space with respect to $|\varphi|$. To do that, we introduce the notion of compass structure generator, that is, of a finite compass structure featuring φ that can be extended to an infinite fulfilling one.

DEFINITION 7. *We say that a finite compass structure $\mathcal{G} = (\mathbb{P}_{\mathbb{O}}, \mathcal{L})$ of size N is* partially fulfilling *if for every point $(x,y) \in \mathbb{P}_{\mathbb{O}}$ such that $y < N-1$, for every relation $R \in \{A, B, \bar{B}, \bar{L}\}$, and for every formula $\psi \in \mathcal{R}eq_R(\mathcal{L}(p))$, one of the following conditions hold:*

1. *there exists a point $p' \in \mathbb{P}_{\mathbb{O}}$ such that $p \; R \; p'$ and $\psi \in \mathcal{O}bs(\mathcal{L}(p'))$ (ψ is fulfilled in p'),*
2. *$R = \bar{B}$ and $\psi \in \mathcal{R}eq_{\bar{B}}(\mathcal{L}(x, N-1))$,*
3. *$R = A$ and $\psi \in \mathcal{R}eq_{\bar{B}}(\mathcal{L}(y, N-1))$,*
4. *$R = \bar{L}$ and $\psi \in \mathcal{R}eq_{\bar{L}}(\mathcal{L}(0,1))$.*

Notice that all B-requests are fulfilled in a partially fulfilling compass structure and that $\bar{B}$, A, and $\bar{L}$ requests are either fulfilled or "transferred to the border" of the compass structure. Moreover, any substructure $\mathcal{G}'$ of a fulfilling compass structure $\mathcal{G}$ is partially fulfilling.

DEFINITION 8. *Given a finite compass structure $\mathcal{G} = (\mathbb{P}_{\mathbb{O}}, \mathcal{L})$ and a row y, a* future witness set *for y is any* minimal *set $\mathcal{F}ut\mathcal{W}it(y) \subseteq \{x : x < y\}$ such that for every $F \in \mathcal{S}hading_{\mathcal{G}}(y)$ there exists a witness $x_F \in \mathcal{F}ut\mathcal{W}it(y)$ that respects the following properties:*

1. *$\mathcal{L}(x_F, y) = F$,*
2. *for every $\psi \in \mathcal{R}eq_{\bar{B}}(F)$ there exists a point $(x_F, y') \in \mathcal{G}$ with $y' > y$ and $\psi \in \mathcal{O}bs(\mathcal{L}(x_F, y))$.*

Since $\mathcal{F}ut\mathcal{W}it(y)$ is minimal, we have that for every $F \in \mathcal{S}hading(y)$ there is exactly one witness x_F in $\mathcal{F}ut\mathcal{W}it(y)$. Hence, $|\mathcal{F}ut\mathcal{W}it(y))| \leqslant 2^{8|\varphi|}$.

DEFINITION 9. *Given a finite compass structure $\mathcal{G} = (\mathbb{P}_{\mathbb{O}}, \mathcal{L})$ and a row y, a* past witness set *for y is any* minimal *set $\mathcal{P}ast\mathcal{W}it(y) \subseteq \mathbb{P}_{\mathbb{O}}$ such that for every request $\psi \in \mathcal{R}eq_{\bar{L}}(\mathcal{O}bs(\mathcal{L}(y-1, y))$ there exists a witness (x_ψ, y_ψ) such that $\psi \in \mathcal{O}bs(\mathcal{L}(x_\psi, y_\psi))$ and $y_\psi < y - 1$.*

Again, by the minimality of $\mathcal{P}ast\mathcal{W}it(y)$ we have that there is at most one distinct point for every $\bar{L}$-formula in $\mathcal{L}(y-1, y)$ and thus $|\mathcal{P}ast\mathcal{W}it(y)| \leqslant |\mathcal{R}eq_{\bar{L}}(y-1, y)| \leqslant |\mathcal{C}l(\varphi)| \leqslant 2 \cdot |\varphi|$.

We concentrate our attention on infinite structures that are unbounded both on the future and on the past (i.e., based on the set of integers $\mathbb{Z}$). The case when the structure is unbounded only in one direction (e.g., the naturals $\mathbb{N}$ or the set of negative integers $\mathbb{Z}^-$) can be tackled in a similar way by appropriately adapting the following notions and theorems.

DEFINITION 10. *Given an $AB\bar{B}\bar{L}$ formula φ and a finite, partially fulfilling compass structure $\mathcal{G} = (\mathbb{P}_{\mathbb{O}}, \mathcal{L})$ of size N, we say that $\mathcal{G}$ is a* compass generator *for φ if there exists four rows y_φ, y_0, y_1, and y_2 which satisfy the following properties:*

G1 *$y_0 < y_1 < y_2$ and $y_0 \leqslant y_\varphi$,*

G2 *$\varphi \in \mathcal{L}(y_\varphi - 1, y_\varphi)$ or $\langle \bar{B} \rangle \varphi \in \mathcal{L}(y_\varphi - 1, y_\varphi)$,*

G3 *$\mathcal{S}hading(y_1) \subseteq \mathcal{S}hading(y_0)$ and $\mathcal{L}(y_0 - 1, y_0) = \mathcal{L}(y_1 - 1, y_1)$,*

G4 *there exists a past witness set $\mathcal{P}ast\mathcal{W}it(y_1)$ such that $y_0 \leqslant \min(\pi_{y_1}(\mathcal{P}ast\mathcal{W}it(y_1)))$,*

G5 $\mathcal{S}hading(N-1) \subseteq \mathcal{S}hading(y_2)$ *and* $\mathcal{L}(y_2-1,y_2) = \mathcal{L}(N-2,N-1)$,

G6 *there exists a future witness set* $\mathcal{F}ut\mathcal{W}it(y_2)$ *for* y_2.

The next theorem shows that the information contained in a compass generator for φ is sufficient to build an infinite fulfilling compass structure featuring φ.

THEOREM 11. *An* $\mathrm{AB\bar{B}\bar{L}}$ *formula* φ *is satisfiable over the integers* $\mathbb{Z}$ *if and only if there exists a compass generator* $\mathcal{G} = (\mathbb{P}_{\mathbb{O}}, \mathcal{L})$ *for* φ.

Proof. **(⇒)** Let φ an $\mathrm{AB\bar{B}\bar{L}}$ formula that is satisfiable over an infinite fulfilling compass structure $\mathcal{G} = (\mathbb{P}_{\mathbb{Z}}, \mathcal{L})$. Since $\mathcal{G}$ features φ we have that there exists a point (x,y) with $\varphi \in \mathcal{L}(x,y)$ and thus the row $y_\varphi = x+1$ respects condition **G2**.

Now, let $\mathcal{I}nf(\mathcal{G})$ be the set of shadings that occurs infinitely often in $\mathcal{G}$. We define y_1 as the greatest row such that for every $y' \leqslant y_1$, $\mathcal{S}hading(y') \in \mathcal{I}nf(\mathcal{G})$, and y_2 as the smallest row such that for every $y' \geqslant y_2$, $\mathcal{S}hading(y') \in \mathcal{I}nf(\mathcal{G})$. Clearly, since $\mathcal{G}$ is unbounded in the past, we can find two rows y_{min} and y_0 such that $y_{min} < y_0$, and a corresponding portion of the grid $\mathbb{P}_{y_{min}} = \{(x,y) : x \geqslant y_{min}\}$ such that *(i)* $y_0 \leqslant y_\varphi$, *(ii)* $y_0 < y_1$, *(iii)* $\mathcal{S}hading(y_1) \subseteq \mathcal{S}hading(y_0)$ in $\mathbb{P}_{y_{min}}$, *(iv)* $\mathcal{L}(y_0-1,y_0) = \mathcal{L}(y_1-1,y_1)$, and *(v)* there exists a past witness set $\mathcal{P}ast\mathcal{W}it(y_1)$ for y_1 such that $y_0 \leqslant \min(\pi_{y_1}(\mathcal{P}ast\mathcal{W}it(y_1)))$ in $\mathbb{P}_{y_{min}}$. Hence, conditions **G3** and **G4** are respected.

Symmetrically, since $\mathcal{G}$ is unbounded in the future, we can find a row $y_{max} > y_2$ and a corresponding portion of the grid $\mathbb{P}^{y_{max}}_{y_{min}} = \{(x,y) : x \geqslant y_{min} \wedge y \leqslant y_{max}\}$ such that

1. $\mathcal{S}hading(y_{max}) \subseteq \mathcal{S}hading(y_2)$,
2. $\mathcal{L}(y_2-1,y_2) = \mathcal{L}(y_{max}-1,y_{max})$, and
3. there exists a future witness set $\mathcal{F}ut\mathcal{W}it(y_2)$ for y_2 in $\mathbb{P}^{y_{max}}_{y_{min}}$.

This shows that conditions **G5** and **G6** are respected as well. Since $y_0 \leqslant y_\varphi$ and $y_0 < y_1 < y_2$ condition **G1** is also respected. Since the restriction of $\mathcal{G}$ to the finite grid $\mathbb{P}^{y_{max}}_{y_{min}}$ is a partially fulfilling compass structure, we have found the required compass generator for φ.

(⇐) Let $\mathcal{G} = (\mathbb{P}_{\mathbb{O}}, \mathcal{L})$ be a compass generator of size N for φ and let $y_0 < y_1 < y_2$ and y_φ be the four rows that satisfy properties **G1**–**G6** of Definition 10. We will define an infinite sequence of partially fulfilling compass structures $\mathcal{G}_0 \subset \mathcal{G}_1 \subset \mathcal{G}_2 \subset \ldots$ such that the infinite union $\mathcal{G}^\omega = \bigcup_{i=0}^{+\infty} \mathcal{G}_i$ is an infinite fulfilling compass structure that features φ. We start from the initial compass structure $\mathcal{G}_0 = (\mathbb{P}^0, \mathcal{L}^0)$ where $\mathbb{P}^0 = \{(x,y) \in \mathbb{P}_{\mathbb{O}} \,:\, x \geqslant y_0 - 1 \wedge y_0 \leqslant y < N\}$ and $\mathcal{L}^0(x,y) = \mathcal{L}(x,y)$ for every point $(x,y) \in \mathbb{P}^0_{\mathbb{O}}$, and we will show how to iteratively build the infinite sequence of compass structures. For every step i of the procedure, let $\mathcal{G}_i = (\mathbb{P}^i, \mathcal{L}^i)$ be the current structure, and let y^i_{min} and y^i_{max} be the minimum and maximum vertical coordinate in $\mathbb{P}^i$, respectively. We guarantee that the following invariant is respected:

(INV) $\mathcal{S}hading_{\mathcal{G}_i}(y^i_{max}) \subseteq \mathcal{S}hading_{\mathcal{G}}(y_2)$,
$\mathcal{S}hading_{\mathcal{G}_i}(y^i_{min} + y_1 - y_0) \subseteq \mathcal{S}hading_{\mathcal{G}}(y_0)$,
$\mathcal{L}^i(y^i_{max}-1, y^i_{max}) = \mathcal{L}(y_2-1,y_2)$, and $\mathcal{L}^i(y^i_{min}-1, y^i_{min}) = \mathcal{L}(y_0-1,y_0)$.

The invariant trivially holds for $\mathcal{G}_0$. Now, suppose that $\mathcal{G}_i$ respects **(INV)** and let $k_{past} = y_1 - y_0$ and $k_{future} = N - y_2$. Figure 3 depicts how $\mathcal{G}_{i+1} = (\mathbb{P}^{i+1}, \mathcal{L}^{i+1})$ can be built from $\mathcal{G}_i$. Formally, the procedure is defined as follows.

a) $y^{i+1}_{min} = y^i_{min} - k_{past}$, $y^{i+1}_{max} = y^i_{max} + k_{future}$, and $\mathbb{P}^{i+1} = \{(x,y) \in \mathbb{P}_{\mathbb{Z}} \,:\, x \geqslant y^{i+1}_{min} - 1 \wedge y^{i+1}_{min} \leqslant y < y^{i+1}_{max}\}$.

b) for every point $p \in \mathbb{P}^{i+1} \cap \mathbb{P}^i$, let $\mathcal{L}^{i+1}(p) = \mathcal{L}^i(p)$.

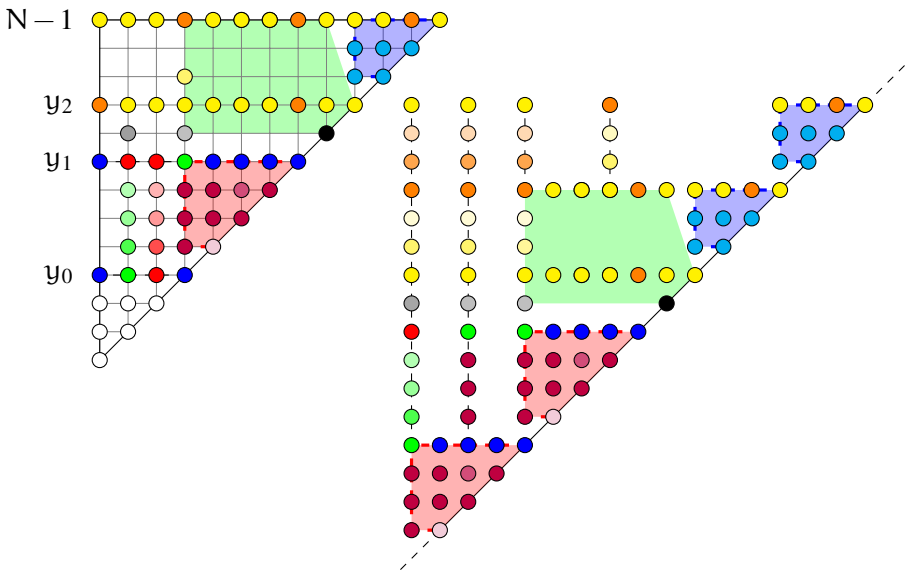

Figure 3: A compass generator (left) and a portion of the generated infinite compass structure (right).

c) for every point $(x,y) \in \mathbb{P}^{i+1} \setminus \mathbb{P}^i$ such that $y \leqslant y^i_{min}$, let $\mathcal{L}^{i+1}(x,y) = \mathcal{L}^i(x + k_{past}, y + k_{past})$ (red area in Fig. 3).

d) for every point $(x,y) \in \mathbb{P}^{i+1} \setminus \mathbb{P}^i$ such that $x \geqslant y^i_{max}$, let $\mathcal{L}^{i+1}(x,y) = \mathcal{L}^i(x - k_{future}, y - k_{future})$ (blue area in Fig. 3).

e) By construction, for every point (x, y^i_{min}) with $x < y^i_{min} - 1$ we have that $\mathcal{L}^{i+1}(x, y^i_{min}) = \mathcal{L}^i(x + k_{past}, y^i_{min} + k_{past})$. Since $\mathcal{G}_i$ respects the invariant, $\mathcal{L}^i(x + k_{past}, y^i_{min} + k_{past}) = \mathcal{L}^{i+1}(x, y^i_{min}) \in \mathcal{S}hading_{\mathcal{G}}(y_0)$. Let $(\bar{x}, y_0)$ be a point on the row y_0 with the same labelling of $\mathcal{L}^{i+1}(x, y^i_{min})$: we define the labelling of all points $(x, y^i_{min} + j)$, with $1 \leqslant j \leqslant k_{past}$, as $\mathcal{L}^{i+1}(x, y^i_{min} + j) = \mathcal{L}(\bar{x}, y_0 + j)$. Now, since $\mathcal{L}(\bar{x}, y_0 + k_{past}) \in \mathcal{S}hading_{\mathcal{G}}(y_1)$ ($y_1 = y_0 + k_{past}$) and $\mathcal{S}hading_{\mathcal{G}}(y_1) \subseteq \mathcal{S}hading_{\mathcal{G}}(y_0)$ (**G3**), we can find a point $(\hat{x}, y_0)$ on the row y_0 with the same labelling of $\mathcal{L}^{i+1}(x, y^i_{min} + k_{past})$ and define the labelling of every point $(x, y^i_{min} + k_{past} \cdot j)$ for every $1 < j \leqslant i + 1$. At the end of this procedure we have labelled all points (x,y) such that $y \leqslant y_1$.

f) For every point (x, y_1), by construction, we have that $\mathcal{L}^{i+1}(x, y_1) \in \mathcal{S}hading_{\mathcal{G}}(y_1)$. Let $(\bar{x}, y_1)$ be a point such that $\mathcal{L}^{i+1}(x, y_1) = \mathcal{L}(\bar{x}, y_1)$. As in the previous case, we define the labelling of all points (x,y), with $y_1 < y \leqslant y_2$ as $\mathcal{L}^{i+1}(x,y) = \mathcal{L}(\bar{x}, y)$. At the end of this step we labelled all points (x,y) such that $y \leqslant y_2$.

g) Now, by construction, for every point (x, y_2) we have that $\mathcal{L}^{i+1}(x, y_2) \in \mathcal{S}hading_{\mathcal{G}}(y_2)$. By condition **G6** of Definition 10, there exists a point $\bar{x} \in \mathcal{F}ut\mathcal{W}it(y_2)$ such that $\mathcal{L}^{i+1}(x, y_2) = \mathcal{L}(\bar{x}, y_2)$. We define $\mathcal{L}^{i+1}(x, y_2 + j) = \mathcal{L}(\bar{x}, y_2 + j)$ for every $1 \leqslant j \leqslant k_{future}$. Since $y_2 + k_{future} = N - 1$ and $\mathcal{S}hading(N-1) \subseteq \mathcal{S}hading(y_2)$ we have that $\mathcal{L}^{i+1}(x, N-1) \in \mathcal{S}hading(y_2)$ (**G5**) and thus we can repeat this procedure iteratively until we have labelled all points (x,y) such that $y \leqslant y^{i+1}_{max}$ and $x < y^i_{min} - 1$.

h) To conclude the procedure, we must define the labelling of points (x,y) such that $x \geqslant y^i_{min} - 1$ and $y \geqslant y^i_{max}$. Note that for every point (x, y^i_{max}) with $x \geqslant y^i_{min} - 1$ we have, by the invariant,

that $\mathcal{S}hading_{\mathcal{G}^i}(y^i_{max}) \subseteq \mathcal{S}hading_{\mathcal{G}}(y_2)$. Then there exists a point $\bar{x} \in \mathcal{F}ut\mathcal{W}it(y_2)$ such that $\mathcal{L}^{i+1}(x, y^i_{max}) = \mathcal{L}(\bar{x}, y_2)$. We define $\mathcal{L}^{i+1}(x, y^i_{max}+j) = \mathcal{L}(\bar{x}, y_2+j)$ for every $1 \leqslant j \leqslant k_{future}$.

It is easy to see that $\mathcal{G}_i$ is a partially fulfilling compass structure that respects the invariant. Moreover, suppose that for some point $p = (x, y) \in \mathbb{P}^i$ and relation $R \in \{A, B, \overline{B}, \overline{L}\}$ there exists $\alpha \in \mathcal{R}eq_R(p)$ that is not fulfilled in $\mathcal{G}_i$. We show that $\mathcal{G}_{i+1}$ fulfills the R-request α for p.

- If $R = A$, since $\mathcal{G}^i$ is partial fulfilling and it is finite we have that the point $p' = (y, y^i_{max})$ is such that $\alpha \in \mathcal{R}eq_{\overline{B}}(\mathcal{L}(p'))$. By step h) of the procedure, and by the definition of future witness set, $\mathcal{G}_{i+1}$ contains a point $p'' = (y, y^i_{max}+j)$ such that $\alpha \in \mathcal{L}^{i+1}(p'')$.
- If $R = B$, by Definition 7 we have all the B-requests in a partial fulfilling compass structure are fulfilled and thus this case connot be given.
- If $R = \overline{B}$ the case is analogous to the case of $R = A$.
- If $R = \overline{L}$, since $\mathcal{G}^i$ is partial fulfilling and it is finite we have that $\alpha \in \mathcal{R}eq_{\overline{L}}(\mathcal{L}(y^i_{min}-1, y^i_{min}))$. By point c) of the construction we have that $\mathcal{L}^i(y^i_{min}-1, y^i_{min}) = \mathcal{L}(y_0-1, y_0) = \mathcal{L}(y_1-1, y_1)$. Hence, by condition **G4** of Definition 10 and by the definition of past witness set, there exists a point $(\bar{x}, \bar{y})$ with $y_0 \leqslant \bar{x} < \bar{y} \leqslant y_1$ such that $\alpha \in \mathcal{L}(\bar{x}, \bar{y})$. By construction we have that $\mathcal{L}(\bar{x}, \bar{y}) = \mathcal{L}^{i+1}(\bar{x} - (i+1) \cdot k_{past}, \bar{y} - (i+1) \cdot k_{past})$ and thus and thus the $\overline{L}$-request α for the point p is fulfilled at step $i+1$ by the point $(\bar{x} - (i+1) \cdot k_{past}, \bar{y} - (i+1) \cdot k_{past})$.

Hence, we can conclude that the infinite compass structure $\mathcal{G}^\omega$ is fulfilling. By condition **G2** of Definition 10 we have that $\mathcal{G}^\omega$ features φ and thus that φ is satisfiable over the integers. □

Theorem 11 shows that satisfiability of a formula over infinite models can be reduced to the existence of a finite compass generator for it. However, it does not give any bound on the size of it. In the following we will show how the techniques exploited in Section 3.1 for finite models can be adapted to obtain a doubly exponential bound on the size of compass generators.

DEFINITION 12. *Given a compass generator $\mathcal{G} = (\mathbb{P}_{\mathbb{O}}, \mathcal{L})$, we say that two rows $y < y'$ are* globally compatible *if and only if the following properties holds:*

1. $\mathcal{L}(y-1, y) = \mathcal{L}(y'-1, y')$ *and* $\mathcal{S}hading_{\mathcal{G}}(y) = \mathcal{S}hading_{\mathcal{G}}(y')$,
2. *for every* $\bar{y} \in \{y_\varphi, y_0, y_1, y_2\}$ *it is not the case that* $y \leqslant \bar{y} \leqslant y'$,
3. *there exists a past witness set* $\mathcal{P}ast\mathcal{W}it(y_1)$ *such that for every point* $(\bar{x}, \bar{y}) \in \mathcal{P}ast\mathcal{W}it(y_1)$ *it is not the case that* $y \leqslant \bar{y} \leqslant y'$;
4. *there exists a future witness set* $\mathcal{F}ut\mathcal{W}it(y_2)$ *such that for every point* $\bar{x} \in \mathcal{F}ut\mathcal{W}it(y_2)$ *and every* $\overline{B}$*-request* $\alpha \in \mathcal{R}eq_{\overline{B}}(\mathcal{L}(\bar{x}, y_2)$ *there is a point* $(\bar{x}, \bar{y})$ *such that* $y_2 < \bar{y}$, $\alpha \in \mathcal{O}bs(\mathcal{L}(\bar{x}, y_2))$ *and it is not the case that* $y \leqslant \bar{y} \leqslant y'$;
5. *there exists a witness set* $\mathcal{W}it(y')$ *for* y' *and an* injective mapping function $w : \pi_{y'}(\mathcal{W}it(y') \cup \mathcal{P}ast\mathcal{W}it(y_1) \cup \mathcal{F}ut\mathcal{W}it(y_2)) \mapsto \{x : x < y\}$, *such that* $\mathcal{L}(x, y') = \mathcal{L}(w(x), y)$, *for every* $x \in \pi_{y'}(\mathcal{W}it(y') \cup \mathcal{P}ast\mathcal{W}it(y_1) \cup \mathcal{F}ut\mathcal{W}it(y_2))$, *and* $w(x) = x$, *for every* $x \in \pi_{y'}(\mathcal{P}ast\mathcal{W}it(y_1)$.

Clearly, two globally compatible rows are compatible. The additional conditions of the definition guarantees that the contraction procedure do not remove "meaningful" parts of the compass generator, like the rows y_φ, y_0, y_1, and y_2 (condition 2) or future and past witnesses (conditions 3 and 4).

LEMMA 13. *Let $\mathcal{G}$ be a compass generator for φ of size N. If there exist two global-compatible rows $0 < y < y' < N$ in $\mathcal{G}$, then there exists a compass generator $\mathcal{G}'$ of size $N' = N - y + y'$ that features φ.*

Proof. We can define a function $f : \{0,...,y\} \to \{0,...,y'\}$ and contract $\mathcal{G}$ to a smaller compass structure $\mathcal{G}'$ in the very same way of Lemma 5. It can be easily proved that the obtained $\mathcal{G}'$ is a partial fulfilling compass structure. Let $k = y' - y$ and let $y'_\varphi = y_\varphi$ if $y_\varphi < y$, $y'_\varphi = y_\varphi - k$ otherwise. To prove that $\mathcal{G}'$ is a compass generator, let us consider the following four cases.

- If $y' < y_0$, then we have that $y'_i = y_i - k$ for $i \in \{0,1,2,\varphi\}$ satisfy conditions **G1-G6** in $\mathcal{G}'$.
- If $y_0 < y < y' < y_1$, then for every point $(\bar{x},\bar{y}) \in \mathcal{PastWit}(y_1)$ we have that either $f(\bar{x},\bar{y}) = (\bar{x},\bar{y})$ (when $\bar{y} < y$) or $f(\bar{x},\bar{y}-k) = (w(\bar{x}),\bar{y}-k) = (\bar{x},\bar{y})$ (when $\bar{y} > y'$), and thus $\mathcal{PastWit}(y_1)$ is a past witness set for $\mathcal{G}'$ as well. From this we can conclude that $y'_\varphi, y_0, y_1 - k$, and $y_2 - k$ satisfy conditions **G1-G6** in $\mathcal{G}'$.
- If $y_0 < y_1 < y < y' < y_2$, then it is easy to prove that y'_φ, y_0, y_1 and $y_2 - k$ satisfy **G1-G6** in $\mathcal{G}'$.
- If $y_0 < y_1 < y_2 < y < y'$, then it is easy to observe that y'_φ, y_0, y_1 and y_2 satisfy **G1-G6** in $\mathcal{G}'$.

Hence, in all possible cases $\mathcal{G}'$ is a compass generator for φ. □

THEOREM 14. *An $AB\bar{B}\bar{L}$-formula φ is satisfied by some infinite interval structure iff it is featured by some compass generator of length $N \leqslant (2|\varphi|+1)^{2^{8|\varphi|}} \cdot 2^{16|\varphi|^2+8|\varphi|}$ (i.e., double exponential in $|\varphi|$).*

Proof. Suppose that φ is satisfied by a infinite interval structure $\mathcal{S}$. By Theorem 11, there is a compass generator $\mathcal{G}$ that features φ. By Lemma 13, we can assume without loss of generality that all rows of $\mathcal{G}$ are pairwise global-incompatible. Let c_y the characteristic function defined in the proof of Theorem 6. Now, let $x_1 < \ldots < x_k$ be the ordered sequence of the points in $\mathcal{PastWit}(y_1)$. We associate to every row y a finite word W_y of length $|W_y| \leqslant k \leqslant 2 \cdot |\varphi|$ on the alphabet $\mathcal{A}_\varphi$ ($|\mathcal{A}_\varphi| = 2^{8|\varphi|}$) such that for every $x_i \in \mathcal{PastWit}(y_1)$, $W(i) = \mathcal{L}(x_i, y)$. It is easy to prove that two rows $y < y'$ in $\mathbb{O}$ with $c_y(F) = c_{y'}$, $W_y = W_{y'}$ and such that $\mathcal{L}(y'-1, y') = \mathcal{L}(y-1, y)$ are global-compatible.

Since the number of possible characteristic functions is bounded by $(2|\varphi|+1)^{2^{8|\varphi|}}$, and the number of possible words is bounded by $(2^{8|\varphi|})^{2\cdot|\varphi|} = 2^{16|\varphi|^2}$, $\mathcal{G}$ cannot have more than $(2|\varphi|+1)^{2^{8|\varphi|}} \cdot 2^{16|\varphi|^2+8|\varphi|}$ rows, and thus N is at most doubly exponential in $|\varphi|$. □

4 Complexity bounds to the satisfiability problem for $AB\bar{B}\bar{L}$

In this section, we discuss the complexity of the satisfiability problem for $AB\bar{B}\bar{L}$ interpreted over strongly discrete interval temporal structures. An EXPSPACE lower bound on the complexity follows from the reduction of the *exponential-corridor tiling problem* (which is known to be EXPSPACE-complete [10]) to the satisfiability problem for the fragment $AB\bar{B}$ given in [8].

To give an upper bound to the complexity we claim that the existence of a compass structure (or compass generator) $\mathcal{G}$ that features a given formula φ can be decided by verifying suitable local (and stronger) consistency conditions over all pairs of contiguous rows, in a similar way to the EXPSPACE algorithm given in [8] for $AB\bar{B}$. In this way, to check those local conditions it is sufficient to store only *(i)* a counter y storing the number of the current row, *(ii)* two guessed shadings S and S' associated with the rows y and $y+1$, and *(iii)* the characteristic functions of the shadings of y and $y+1$. Since all this information needs only an exponential amount of space, the complexity of the satisfiability problem for $AB\bar{B}\bar{L}$ is in EXPSPACE. Summing up, we obtain the following tight complexity result.

THEOREM 15. *The satisfiability problem for* $\mathrm{AB\bar{B}\bar{L}}$ *interpreted strongly complete linear orders is EXPSPACE-complete.*

5 Conclusions

We considered an interval temporal logic ($\mathrm{AB\bar{B}\bar{L}}$) with four modalities, corresponding, respectively, to Allen's interval relations *meets*, *begins*, *begun-by*, and *before*, and interpreted in the class of all strongly discrete linearly ordered sets, which includes, among others, all frames built over $\mathbb{N}$, $\mathbb{Z}$, and finite orders. We showed that this logic is decidable in EXPSPACE, and complete for this class. The importance of this result relies on the fact that, for the considered interpretations, this logic is maximal with respect to decidability. Moreover, these results represent a non-trivial contribution towards the complete classification of all fragments of Halpern and Shoham's modal logic of intervals. We plan to complete the study of this particular language when it is interpreted over other classes of orders, such as the class of all dense linearly ordered sets, or the class of all linear orders, and to refine these results to include point-intervals, too.

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