

# Metric Propositional Neighborhood Logics: expressiveness, decidability, and undecidability

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**Abstract.** Interval temporal logics formalize reasoning about interval structures over (usually) linearly ordered domains, where time intervals are the primitive ontological entities and truth of formulae is defined relative to time intervals, rather than time instants. In this paper, we introduce and study Metric Propositional Neighborhood Logic (MPNL) over natural numbers. MPNL features two modalities referring, respectively, to an interval that is “met by” the current one and to an interval that “meets” the current one, plus an infinite set of length constraints, regarded as atomic propositions, to constrain the lengths of intervals. We argue that MPNL can be successfully used to capture important concepts and scenarios in different areas of artificial intelligence combining qualitative and quantitative interval temporal reasoning, thus providing a viable alternative to well-established logical frameworks such as Duration Calculus. We show that MPNL is decidable in double exponential time and that it is expressively complete with respect to a well-defined subfragment of the two-variable fragment  $\text{FO}^2[\mathbb{N}, =, <, s]$  of first-order logic for linear orders with successor function, interpreted over natural numbers. Moreover, we show that MPNL can be extended in a natural way to cover full  $\text{FO}^2[\mathbb{N}, =, <, s]$ , but, unexpectedly, the latter (and hence the former), turns out to be undecidable.

## 1 Introduction

Interval temporal logics provide a natural framework for temporal reasoning about interval structures over linearly (or partially) ordered domains. They take time intervals as the primitive ontological entities and define truth of formulae relative to time intervals, rather than time instants. Interval logics feature modal operators that correspond to various relations between pairs of intervals. In particular, the logic HS, introduced by Halpern and Shoham in [17], features a set of modal operators that makes it possible to express all Allen’s interval relations [1].

Interval-based formalisms have been extensively used in various areas of AI, such as, for instance, planning and plan validation, theories of action and change, natural language processing, and constraint satisfaction problems. However, most of them make severe syntactic and semantic restrictions that considerably weaken their expressive power. Interval temporal logics relax these restrictions, thus allowing one to cope with much more complex application domains and scenarios. Unfortunately, many of them, including HS and the majority of its fragments, turn out to be undecidable (a comprehensive survey

can be found in [7]). One of the few cases of decidable interval logic with truly interval semantics, that is, not reducible to point-based semantics, is Propositional Neighborhood Logic (PNL), interpreted over various classes of interval structures (all, dense, and discrete linear orders, integers, natural numbers) [16]. PNL is a fragment of HS with only two modalities, corresponding to Allen’s relations *meets* and its inverse *met by*.

In this paper, we consider a proper extension of PNL over natural numbers, called *Metric PNL* (MPNL), that features a family of special atomic propositions representing integer constraints (equalities and inequalities) on the length of the intervals over which they are evaluated. The uni-modal right-neighborhood fragment of MPNL has recently been introduced and studied in [9] – the main precursor of this paper, which extends and strengthens it substantially. MPNL is particularly suitable for quantitative interval reasoning, and thus it emerges as a viable alternative to existing logical systems for quantitative temporal reasoning. Various metric extensions to point-based temporal logics have been proposed in the literature. They include Timed Propositional Temporal Logic (TPTL), developed by Alur and Henzinger in [2], Montanari and de Rijke’s two-sorted metric temporal logics [20], Hirshfeld and Rabinovich’s Quantitative Monadic Logic of Order [18], and Owakine and Worrell’s Metric Temporal Logic [22]. Little work has been done in the interval logic setting. Among the few contributions, we mention the extension of Allen’s Interval Algebra with a suitable notion of distance developed by Kautz and Ladkin in [19]. The most important quantitative interval temporal logic is Duration Calculus (DC) [11], which is quite expressive, but undecidable. A number of variants and fragments of DC have been proposed in the literature to model and reason about real-time processes and systems [5, 11, 12, 13]. Many of them recover decidability by imposing semantic restrictions, such as the *locality* principle, that essentially reduce the interval system to a point-based one.

The main results of the present paper are: (i) decidability and complexity of the satisfiability problem for MPNL (Section 4); (ii) expressive completeness of MPNL with respect to a well-defined subfragment of  $\text{FO}^2[\mathbb{N}, =, <, s]$  (Section 5); (iii) an extension of MPNL which is expressively complete with respect to full  $\text{FO}^2[\mathbb{N}, =, <, s]$  and proof of their undecidability (Section 5).

## 2 MPNL over Natural Numbers

Given a linearly ordered domain  $\mathbb{D} = \langle D, < \rangle$ , interpreted as the set of natural numbers  $\mathbb{N}$  or any finite subset of it, a (*non-strict*) interval over  $\mathbb{D}$  is any ordered pair  $[i, j]$  such that  $i \leq j$ . An *interval structure* is a pair  $\langle \mathbb{D}, \mathbb{I}(\mathbb{D}) \rangle$ , where  $\mathbb{I}(\mathbb{D})$  is the set of all intervals over  $\mathbb{D}$ . An *interval model* is a tuple  $M = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), V \rangle$ , where  $\langle \mathbb{D}, \mathbb{I}(\mathbb{D}) \rangle$  is

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an interval structure and  $V : \mathcal{AP} \rightarrow 2^{\mathbb{D}}$  is a valuation function assigning to every atomic proposition the set of intervals over which it holds. We define the standard *distance* function  $\delta : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  as  $\delta(i, j) = |i - j|$  (notice that natural numbers appear both as points in the interval structure and as interval lengths). As a matter of fact, all results we will provide may be suitably rephrased for any function  $\delta$  satisfying the standard properties of distance over a linear order.

To add a metric dimension to PNL, we introduce a set of special propositional letters referring to the length of the current interval. They can be viewed as a metric generalization of the modal constant  $\pi$  of PNL that ‘identifies’ intervals of the form  $[i, i]$  [16]. Formally, for every  $k \in \mathbb{N}$  and  $C \in \{<, \leq, =, \geq, >\}$ , we define the length constraint  $\text{len}_{Ck}$ . The *formulae* of MPNL, denoted by  $\varphi, \psi, \dots$ , are generated by the following grammar:

$$\varphi ::= \text{len}_{Ck} \mid p \mid \neg\varphi \mid \varphi \vee \varphi \mid \diamond_r\varphi \mid \diamond_l\varphi.$$

Given an interval model  $M = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), V \rangle$  and an interval  $[i, j]$  over it, the semantics of MPNL-formulae is given by the clauses:

- $M, [i, j] \models \text{len}_{Ck}$  iff  $\delta(i, j)Ck$ ;
- $M, [i, j] \models p$  iff  $p \in V([i, j])$  for any  $p \in \mathcal{AP}$ ;
- $M, [i, j] \models \neg\psi$  iff it is not the case that  $M, [i, j] \models \psi$ ;
- $M, [i, j] \models \psi \vee \tau$  iff  $M, [i, j] \models \psi$  or  $M, [i, j] \models \tau$ ;
- $M, [i, j] \models \diamond_r\psi$  iff there exists  $h \geq j$  such that  $M, [j, h] \models \psi$ ;
- $M, [i, j] \models \diamond_l\psi$  iff there exists  $h \leq i$  such that  $M, [h, i] \models \psi$ .

A MPNL-formula  $\varphi$  is *satisfiable* if there exist a model  $M$  and an interval  $[b, e]$  over it such that  $M, [b, e] \models \varphi$ . We can limit ourselves to consider only length constraints of the form  $\text{len}_{=k}$ , as all the remaining ones can be defined in terms of them.

### 3 MPNL at Work

Finding an appropriate balancing between expressive power and computational complexity is a challenge for every knowledge representation and reasoning formalism. Interval temporal logics are not an exception in this respect. We believe that MPNL features a good compromise between the two requirements. In the following, we show that MPNL makes it possible to encode metric versions of basic operators of point-based linear temporal logic (LTL) as well as of interval modalities corresponding to Allen’s relations; in addition, we show that it allows one to express limited forms of fuzziness.

First, MPNL is expressive enough to encode the strict *sometimes in the future* (resp., *sometimes in the past*) operator of LTL:

$$\diamond_r(\text{len}_{>0} \wedge \diamond_r(\text{len}_{=0} \wedge p)). \quad (1)$$

Moreover, length constraints allow one to define a metric version of the *until* (resp., *since*) operator. For instance, the condition: ‘*p is true at a point in the future at distance k from the current interval and, until that point, q is true (pointwise)*’ can be expressed as follows:

$$\diamond_r(\text{len}_{=k} \wedge \diamond_r(\text{len}_{=0} \wedge p)) \wedge \square_r(\text{len}_{<k} \rightarrow \diamond_r(\text{len}_{=0} \wedge q)). \quad (2)$$

MPNL can also be used to constrain interval length and to express basic interval relations. First, we can easily constrain the length of the intervals over which a given property holds to be at least (resp., at most, exactly)  $k$ . As an example, the following formula constrains  $p$  to hold only over intervals of length greater than  $k$  and less than  $k'$ :

$$[G](p \rightarrow \text{len}_{\geq k} \wedge \text{len}_{\leq k'}), \quad (3)$$

where the *universal modality*  $[G]$  is defined as in [16]. By exploiting such a capability, a metric version of all, but one (the ‘*during*’ relation), Allen’s relations can be expressed. As an example, we can state that: ‘*p holds over an interval of bounded length l, with  $k \leq l \leq k'$ , that begins an interval (of arbitrary length) over which q holds*’:

$$p_{\geq k}^{\leq k'} \wedge [G] \bigwedge_{i=k}^{k'} (p \wedge \text{len}_{=i} \rightarrow \diamond_l \diamond_r(\text{len}_{>i} \wedge q)), \quad (4)$$

where  $p_{\geq k}^{\leq k'}$  is a shorthand for (3). As another example, Allen’s relation *contains* can be expressed by pairing  $p_{\geq k}^{\leq k'}$  with:

$$[G] \bigwedge_{i=k}^{k'} (p \wedge \text{len}_{=i} \rightarrow \bigvee_{j, j', j+j' < i} (\diamond_l \diamond_r(\text{len}_{=j} \wedge \diamond_r(\text{len}_{=j'} \wedge q)))). \quad (5)$$

Finally, length constraints allow one to express some form of ‘fuzziness’. As an example, the condition: ‘*p is true over the current interval and q is true over some interval close to it*’, where by ‘close’ we mean that the right endpoint of the  $p$ -interval is at distance at most  $k$  from the left endpoint of the  $q$ -interval, can be expressed as follows:

$$p \wedge (\diamond_r \diamond_l(\text{len}_{<k} \wedge \diamond_l \diamond_r q) \vee \diamond_r(\text{len}_{<k} \wedge \diamond_r q)). \quad (6)$$

MPNL capabilities suffice to cope with various application domains. As a source of exemplification, we show how to express some basic safety requirements of the classical *gas-burner example* (a formalization of such an example in DC can be found in [11]). Let the propositional letter *Gas* (resp., *Flame*, *Leak*) be used to state that gas is flowing (resp., burning, leaking), e.g.,  $M, [i, j] \models \text{Gas}$  means that gas is flowing over the interval  $[i, j]$ . The formula

$$[G](\text{Leak} \leftrightarrow \text{Gas} \wedge \neg \text{Flame}) \quad (7)$$

states that *Leak* holds over an interval if and only if gas is flowing and not burning over that interval. The condition: ‘*it never happens that gas is leaking for more than k time units*’ can be expressed as:

$$[G](\neg(\text{len}_{>k} \wedge \text{Leak})). \quad (8)$$

Similarly, the condition: ‘*the gas burner will not leak for k time units after the last leakage*’ can be formalized as:

$$[G](\text{Leak} \rightarrow \neg \diamond_l(\text{len}_{<k} \wedge \diamond_l \text{Leak})). \quad (9)$$

We conclude the section by mentioning two application domains where MPNL features are well-suited, namely, medical guidelines and ambient intelligence. In the area of medical guidelines [23], events with duration, e.g., ‘*running a fever*’, possibly paired with metric constraints, e.g., ‘*if a patient is running a fever for more than k time units, then administrate him/her drug D*’, are quite common. In general, many relevant phenomena are inherently interval-based as there are no general rules to deduce their occurrence from point-based data. The use of temporal logic in ambient intelligence, specifically in the area of Smart Homes [3, 4, 15], has been advocated by Combi et al. in [14]. MPNL can be successfully used to express safety requirements referring to situations that can be properly modeled only in terms of time intervals, e.g., ‘*being in the kitchen*’.

### 4 Decidability of MPNL

In this section, we use a model-theoretic argument to show that the satisfiability problem for MPNL has a bounded-model property with

respect to finitely presentable ultimately periodic models, and it is therefore decidable. For lack of space, we only sketch the proofs of the main technical results. From now on, let  $\varphi$  be any MPNL-formula and let  $\mathcal{AP}$  be the set of propositional letters of the language.

**Definition 1** The closure of  $\varphi$  is the set  $CL(\varphi)$  of all subformulae of  $\varphi$  and their negations (we identify  $\neg\neg\psi$  with  $\psi$ ). The set of temporal requests from  $CL(\varphi)$  is the set  $TF(\varphi)$  of all temporal formulae in  $CL(\varphi)$ , that is,  $TF(\varphi) = \{\diamond_r\psi, \diamond_l\psi, \square_r\psi, \square_l\psi \mid \diamond_r\psi, \diamond_l\psi, \square_r\psi, \square_l\psi \in CL(\varphi)\}$ .

**Definition 2** A  $\varphi$ -atom is a set  $A \subseteq CL(\varphi)$  such that for every  $\psi \in CL(\varphi)$ ,  $\psi \in A$  iff  $\neg\psi \notin A$  and for every  $\psi_1 \vee \psi_2 \in CL(\varphi)$ ,  $\psi_1 \vee \psi_2 \in A$  iff  $\psi_1 \in A$  or  $\psi_2 \in A$ .

We denote the set of all  $\varphi$ -atoms by  $A_\varphi$ . One can easily prove that  $|CL(\varphi)| \leq 2^{|\varphi|}$ ,  $|TF(\varphi)| \leq 2^{(|\varphi| - 1)}$ , and  $|A_\varphi| \leq 2^{|\varphi|}$ . We now introduce a suitable labeling of interval structures based on  $\varphi$ -atoms.

**Definition 3** A  $(\varphi)$ -labeled interval structure (LIS for short) is a structure  $\mathbf{L} = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{L} \rangle$ , where  $\langle \mathbb{D}, \mathbb{I}(\mathbb{D}) \rangle$  is the interval structure over natural numbers (or over a finite subset of it) and  $\mathcal{L} : \mathbb{I}(\mathbb{D}) \rightarrow A_\varphi$  is a labeling function such that for every pair of neighboring intervals  $[i, j], [j, r] \in \mathbb{I}(\mathbb{D})$ , if  $\square_r\psi \in \mathcal{L}([i, j])$ , then  $\psi \in \mathcal{L}([j, r])$ , and if  $\square_l\psi \in \mathcal{L}([j, r])$ , then  $\psi \in \mathcal{L}([i, j])$ .

Notice that every interval model  $M$  is a LIS, where the labeling function is the valuation function:

$$\psi \in \mathcal{L}([i, j]) \text{ iff } M, [i, j] \models \psi.$$

Thus, LIS can be thought of as *quasi-models* for  $\varphi$ , in which the truth of formulae containing neither  $\diamond_r$ ,  $\diamond_l$  nor length constraints is determined by the labeling (due to the definitions of  $\varphi$ -atom and LIS). To obtain a model, we must also guarantee that the truth of the other formulae is in accordance with the labeling. To this end, we introduce the notion of fulfilling LIS.

**Definition 4** A LIS  $\mathbf{L} = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{L} \rangle$  is fulfilling iff:

- for every length constraint  $\text{len}_{=k} \in CL(\varphi)$  and every interval  $[i, j] \in \mathbb{I}(\mathbb{D})$ ,  $\text{len}_{=k} \in \mathcal{L}([i, j])$  iff  $\delta(i, j) = k$ ;
- for every temporal formula  $\diamond_r\psi \in TF(\varphi)$  and every interval  $[i, j] \in \mathbb{I}(\mathbb{D})$ , if  $\diamond_r\psi \in \mathcal{L}([i, j])$ , then there exists  $r \geq j$  such that  $\psi \in \mathcal{L}([j, r])$ ;
- for every temporal formula  $\diamond_l\psi \in TF(\varphi)$  and every interval  $[i, j] \in \mathbb{I}(\mathbb{D})$ , if  $\diamond_l\psi \in \mathcal{L}([i, j])$ , then there exists  $r \leq i$  such that  $\psi \in \mathcal{L}([r, i])$ .

Clearly, every interval model is a fulfilling LIS. Conversely, every fulfilling LIS  $\mathbf{L} = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{L} \rangle$  can be transformed into a model  $M(\mathbf{L})$  by defining the valuation in accordance with the labeling. Then, one can prove that for every  $\psi \in CL(\varphi)$  and interval  $[i, j] \in \mathbb{I}(\mathbb{D})$ ,  $\psi \in \mathcal{L}([i, j])$  iff  $M(\mathbf{L}), [i, j] \models \psi$  by a routine induction on  $\psi$ .

**Definition 5** Given a LIS  $\mathbf{L} = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{L} \rangle$  and an interval  $[i, j] \in \mathbb{I}(\mathbb{D})$ , the set of left temporal requests at  $i$  (resp., right temporal requests at  $j$ ), denoted by  $REQ^L(i)$  (resp.,  $REQ^R(j)$ ), is the set of temporal formulae of the forms  $\diamond_l\varphi$ ,  $\square_l\varphi$  (resp.,  $\diamond_r\varphi$ ,  $\square_r\varphi$ ) in  $TF(\varphi)$  belonging to the labeling of any interval beginning in  $i$  (resp., ending in  $j$ ). Given a point  $r \in D$ , the set  $REQ(r)$  is defined as  $REQ(r) = REQ^L(r) \cup REQ^R(r)$ ,

We denote by  $REQ(\varphi)$  the set of all possible sets of temporal requests over  $CL(\varphi)$ . It is easy to show that  $|REQ(\varphi)| = 2^{\frac{|TF(\varphi)|}{2}}$ . From now on, we will use the symbol  $m$  for  $\frac{|TF(\varphi)|}{2}$  and  $k$  for the maximum among all natural numbers occurring in the length constraints in  $\varphi$ . For example, if  $\varphi = \diamond_r(\text{len}_{>3} \wedge p \rightarrow \diamond_l(\text{len}_{>5} \wedge q))$ , then  $m = 2$  and  $k = 5$ . It is easy to observe that given any set of temporal requests  $REQ^R(j)$  (resp.,  $REQ^L(i)$ ), all formulae in it can be satisfied using at most  $m$  different points  $r$  such that  $r > j$  (resp.,  $r < i$ ).

Now, consider any MPNL-formula  $\varphi$  such that  $\varphi$  is satisfiable on a finite model. We have to show that we can restrict our attention to models with a bounded dimension.

**Definition 6** Given any LIS  $\mathbf{L} = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{L} \rangle$ , we say that a  $(k)$ -sequence in  $\mathbf{L}$  is a sequence of  $(k)$  consecutive points in  $D$ . Given a sequence  $\sigma$  in  $\mathbf{L}$ , its sequence of requests  $REQ(\sigma)$  is defined as the sequence of temporal requests at the points in  $\sigma$ . We say that  $i \in \mathbf{L}$  starts a  $k$ -sequence  $\sigma$  if the temporal requests at  $i, \dots, i + k - 1$  form an occurrence of  $REQ(\sigma)$ . Moreover, the sequence of requests  $REQ(\sigma)$  is said to be abundant in  $\mathbf{L}$  (on an interval  $[i, j]$ ) iff it has at least  $2 \cdot (m^2 + m) \cdot |REQ(\varphi)| + 1$  disjoint occurrences in  $D$  (in the interval  $[i, j]$ ).

**Lemma 7** Let  $\mathbf{L} = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{L} \rangle$  be any LIS such that  $REQ(\sigma)$  is abundant in it. Then, there exists an index  $q$  such that for each element  $\mathcal{R} \in \{REQ(d) \mid i_q < d < i_{q+1}\}$ , where  $i_q$  and  $i_{q+1}$  begin the  $q$ -th and the  $q + 1$ -th occurrence of  $\sigma$ , respectively,  $\mathcal{R}$  occurs at least  $m^2 + m$  times before  $i_q$  and at least  $m^2 + m$  times after  $j_{q+1}$ .

**Lemma 8** Let  $\mathbf{L} = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{L} \rangle$  be a fulfilling LIS that satisfies  $\varphi$ . Suppose that there exist an abundant  $k$ -sequence of requests  $REQ(\sigma)$  and let  $q$  be the index whose existence is guaranteed by Lemma 7. Then, there exists a fulfilling LIS  $\bar{\mathbf{L}} = \langle \bar{\mathbb{D}}, \bar{\mathbb{I}}(\bar{\mathbb{D}}), \bar{\mathcal{L}} \rangle$  that satisfies  $\varphi$  such that  $\bar{D} = D \setminus \{i_q, \dots, i_{q+1} - 1\}$ .

**Proof.** [sketch] Let  $\mathbf{L} = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{L} \rangle$  be a fulfilling LIS satisfying  $\varphi$  at some  $[b, e]$ ,  $REQ(\sigma)$  be an abundant  $k$ -sequence in  $\mathbf{L}$ , and  $q$  be the index identified by Lemma 7. Moreover, let  $D^- = \{i_q, \dots, i_{q+1} - 1\}$  and  $D' = D \setminus D^-$ . We denote by  $\mathbb{I}(D')$  the set of all intervals over  $D'$ . We have the problem of suitably re-defining the evaluation of all intervals on  $D'$  in a way preserving the temporal requests at all points in  $D'$  and still satisfying  $\varphi$ .

First, we consider all points  $d < i_q$  and for each of them, for all  $p$  such that  $0 \leq p < k - 1$ , we put  $\mathcal{L}'([d, i_{q+1} + p]) = \mathcal{L}([d, i_q + p])$ . In such a way, we guarantee that the intervals whose length has been shortened as an effect of the elimination of the points in  $D^-$  have a correct labeling in terms of all length constraints of the forms  $\text{len}_{=k'}$  and  $\neg\text{len}_{=k'}$ , with  $k' \leq k$ . Moreover, since the requests (in both directions) in  $\mathbf{L}$  at  $i_{q+1} + p$  are equal to the requests at  $i_q + p$ , this operation is safe with respect to universal and existential requirements. Finally, since the lengths of intervals beginning before  $i_q$  and ending after  $i_{q+1}$  are greater than  $k$  both in  $\mathbf{L}$  and in  $\mathbf{L}'$ , there is no need to change their labeling.

The structure  $\mathbf{L}' = \langle D', \mathbb{I}(D'), \mathcal{L}' \rangle$  defined so far is obviously a LIS, but it is not necessarily a fulfilling one. The removal of the points in the set  $D^-$  may generate *defects*, that is, situations in which there exists a point  $d < i_q$  (resp.,  $d \geq i_{q+1}$ ) and a formula of the type  $\diamond_r\psi$  (resp.,  $\diamond_l\psi$ ) belonging to  $REQ(d)$ , which was satisfied on  $[d, d']$  (resp.,  $[d', d]$ ), with  $d' \in D^-$ , and it is not satisfied anymore. In order to repair such defects, one can simply redefine the labels at intervals starting at  $d$  and ending at some (eliminated)  $d'$

using the  $m^2 + m$  ‘copies’ of  $d'$  that, by hypothesis, are in  $D'$  (as already pointed out, at most  $m$  points are needed to satisfy all requests at  $d'$ ). This construction is similar to the one used in [9] to show that the single-modality metric PNL has the small-model property. If we repeat such a procedure sufficiently many times, we obtain a finite sequence of LIS, the last one of which is the required  $\bar{\mathbf{L}}$ . ■

The lemma above guarantees that we can eliminate sequences of requests that occur ‘sufficiently many’ times in a LIS, without ‘spoiling’ the LIS. This eventually allows us to prove the following small model theorem for finite satisfiability of MPNL.

**Theorem 9 (Small Model Theorem)** *If  $\varphi$  is any finitely satisfiable formula of MPNL, then there exists a fulfilling, finite LIS  $\mathbf{L} = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{L} \rangle$  that satisfies  $\varphi$  such that  $|D| \leq |\text{REQ}(\varphi)|^k \cdot (2 \cdot (m^2 + m) \cdot |\text{REQ}(\varphi)| + 1) \cdot k + k - 1$ .*

To deal with formulae that are satisfiable only over infinite models, we need to introduce a finite (periodic) representation for them and, then, to show that we are able to bound the length of the prefix and the period.

**Definition 10** *A LIS  $\mathbf{L} = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{L} \rangle$  is ultimately periodic, with prefix  $L$ , period  $P$ , and threshold  $k$  if:*

- for every interval  $[i, j]$  such that  $i \geq L$ ,  $\mathcal{L}([i, j]) = \mathcal{L}([i + P, j + P])$ ;
- for every interval  $[i, j]$  such that  $j \geq L$  and  $\delta(j, i) > k$ ,  $\mathcal{L}([i, j]) = \mathcal{L}([i, j + P])$ .

It is worth noticing that, in every ultimately periodic LIS,  $\text{REQ}(i) = \text{REQ}(i + P)$ , for  $i \geq L$ , and that every ultimately periodic LIS is finitely presentable: it suffices to define its labeling only on the intervals  $[i, j]$  such that  $j \leq L + P + \max(k, P)$ ; thereafter, it can be uniquely extended by periodicity. Furthermore, we can identify a finite LIS with an ultimately periodic one with a period  $P = 0$ .

**Lemma 11** *Let  $\mathbf{L} = \langle \mathbb{N}, \mathbb{I}(\mathbb{N}), \mathcal{L} \rangle$  be an infinite fulfilling LIS over  $\mathbb{N}$  that satisfies a formula  $\varphi$  on  $[b, e]$  for some  $b, e \in \mathbb{N}$ . Then, there exists an infinite ultimately periodic fulfilling LIS  $\bar{\mathbf{L}} = \langle \mathbb{N}, \mathbb{I}(\mathbb{N}), \bar{\mathcal{L}} \rangle$  over  $\mathbb{N}$  that satisfies  $\varphi$  on  $[b, e]$ .*

**Proof.** [sketch] Let  $[b, e]$  be an interval such that  $\varphi \in \mathcal{L}([b, e])$ . We define the set  $\text{REQ}_{inf}(\mathbf{L})$  as the subset of  $\text{REQ}(\varphi)$  containing all and only the sets of requests that occur infinitely often in  $\mathbf{L}$ . We can choose two points  $L, M$ , with  $L + k < M$  such that  $L, M$  are the least points in  $\mathbb{N}$  that satisfy the following conditions: (i)  $L \geq e$ ; (ii) for each point  $r \geq L$ ,  $\text{REQ}(r) \in \text{REQ}_{inf}(\mathbf{L})$ ; (iii) every set of requests  $\mathcal{R} \in \text{REQ}_{inf}(\mathbf{L})$  occurs at least  $m^2 + m$  times before  $L$  and it occurs at least  $m^2 + m$  times between  $L$  and  $M$ ; (iv) for each point  $i < L$  and any formula  $\diamond_r \psi \in \text{REQ}(i)$ ,  $\psi$  is satisfied over some interval  $[i, j]$ , with  $j < M$ ; and (v) the  $k$ -sequences of requests starting at  $L$  and at  $M$  are the same.

We put  $P = M - L$ . We can build an infinite ultimately periodic structure  $\bar{\mathbf{L}}$  over the natural numbers with prefix  $L$ , period  $P$ , and threshold  $k$ . To this end, for all points  $d < M$ , we put  $\bar{\text{REQ}}(d) = \text{REQ}(d)$  and, for all points  $M + n$ , with  $0 \leq n < P$ , we put  $\bar{\text{REQ}}(M + n) = \text{REQ}(L + n)$ . The labeling can now be defined as follows. For all intervals  $[i, j]$  such that  $j < M$ , we put  $\bar{\mathcal{L}}([i, j]) = \mathcal{L}([i, j])$ . As for any interval  $[i, j]$ , with  $M \leq j < M + P$ , (a) if  $i \geq M$ , we put  $\bar{\mathcal{L}}([i, j]) = \mathcal{L}([i - P, j - P])$ , (b) if  $i < M$  and  $\delta(i, j - P) > k$ , we put  $\bar{\mathcal{L}}([i, j]) = \mathcal{L}([i, j - P])$ , and (c) if

$i < M$  and  $\delta(i, j - P) \leq k$ , we put  $\bar{\mathcal{L}}([i, j]) = \mathcal{L}([l, h])$ , where  $l$  and  $h$  are such that  $\bar{\text{REQ}}(i) = \text{REQ}(l)$ ,  $\bar{\text{REQ}}(j) = \text{REQ}(h)$ , and  $\delta(h, l) > k$ . Existence of  $h$  and  $l$  are guaranteed by conditions (i)–(v). This construction labels all subintervals in  $[0, M + P]$  in such a way that  $\bar{\mathbf{L}}$  is a LIS, but not necessarily a fulfilling one. As a matter of fact, there could exist points  $L \leq i \leq M$  such that a formula  $\diamond_r \psi \in \bar{\text{REQ}}(i)$  is not fulfilled anymore in  $\bar{\mathbf{L}}$ . To fix such defects, one can proceed as in the proof of Lemma 8, exploiting the conditions (i)–(v). Finally,  $\bar{\mathbf{L}}$  can be extended over  $\mathbb{I}(\mathbb{N})$  in a unique, ultimately periodic and “fulfillness-preserving” way. ■

**Theorem 12 (Small Periodic Model Theorem)** *If  $\varphi$  is any satisfiable formula of MPNL, then there exists a fulfilling, ultimately periodic LIS satisfying  $\varphi$  such that both the length  $L$  of the prefix and the length  $P$  of the period are less or equal to  $|\text{REQ}(\varphi)|^k \cdot (2 \cdot (m^2 + m) \cdot |\text{REQ}(\varphi)| + 1) \cdot k + k - 1$ .*

**Proof.** Existence of an ultimately periodic fulfilling LIS is guaranteed by Lemma 11. The bound on the prefix and of the period can be proved by exploiting Lemma 8.

**Corollary 13** *The satisfiability problem for MPNL, interpreted over  $\mathbb{N}$ , is decidable.*

The results of the previous section immediately give a double exponential time nondeterministic procedure for checking the satisfiability of any MPNL-formula  $\varphi$ . Such a procedure non-deterministically checks models whose size, in general, is  $O(2^{|\varphi|^k})$ , where  $|\varphi|$  is the length of the formula to be checked for satisfiability. It has been shown in [9] that, in the case in which  $k$  is represented in binary, the one-modality fragment of MPNL is complete for EX-SPACE. This means that the complexity for MPNL, in the general case, is located between EXSPACE and 2NEXPTIME (whereas the exact complexity is still an open problem). It is worth noticing that when  $k$  is a constant it does not influence the complexity class and so, since we have a  $\text{NTIME}(2^{|\varphi|})$  procedure for satisfiability and a NEXPTIME-hardness result [10], we can conclude that MPNL is NEXPTIME-complete. Similarly, when  $k$  is expressed in unary, the value of  $k$  increases linearly with the length of the formula and, thus,  $\text{NTIME}(2^{k|\varphi|}) = \text{NTIME}(2^{|\varphi|^2})$ ; therefore, as in the previous case, MPNL is NEXPTIME-complete.

## 5 Expressive Completeness and Undecidable Extensions

Let us denote by  $\text{FO}^2[=]$  the fragment of first-order logic with equality whose language contains only two distinct variables; we can further assume w.l.o.g. that the arity of every relation in the considered vocabulary is exactly 2 (since atoms in the two-variable fragment can involve at most two distinct variables). We denote its formulae by  $\alpha, \beta, \dots$ . For example, the formula  $\forall x(P(x) \rightarrow \forall y \exists x Q(x, y))$  belongs to  $\text{FO}^2$ , while the formula  $\forall x(P(x) \rightarrow \forall y \exists z(Q(z, y) \wedge Q(z, x)))$  does not. The logic  $\text{FO}^2[\mathbb{N}, =, <]$  is the extension of  $\text{FO}^2$ , interpreted over natural numbers, over a purely relational vocabulary  $\{=, <, P, Q, \dots\}$  including equality and a distinguished binary relation  $<$  interpreted as the standard linear ordering. Decidability (NEXPTIME-completeness) of  $\text{FO}^2[\mathbb{N}, =, <]$  has been shown in [21]. In [8], it has been shown that  $\text{FO}^2[\mathbb{N}, =, <]$  is expressively complete with respect to PNL. For the comparison of these logics suitable truth-preserving model transformations between interval models and relational models have been defined.

**Table 1.** Translation clauses from  $\text{FO}_r^2[\mathbb{N}, =, <, s]$  to MPNL.

$\tau[x, y](s^k(z) = s^m(z)) = \top (z \in \{x, y\}), \text{ if } k = m$	$\tau[x, y](P(s^k(x), s^m(x))) =$
$\tau[x, y](s^k(z) = s^m(z)) = \perp (z \in \{x, y\}), \text{ if } k \neq m$	$\diamond_l \diamond_r (\text{len}_{=k} \wedge \diamond_r (\text{len}_{=m-k} \wedge p^{\leq})), \text{ if } k < m$
$\tau[x, y](s^k(z) < s^m(z)) = \perp (z \in \{x, y\}), \text{ if } k \geq m$	$\tau[x, y](P(s^k(x), s^m(x))) =$
$\tau[x, y](s^k(z) < s^m(z)) = \top (z \in \{x, y\}), \text{ if } k < m$	$\diamond_l \diamond_r (\text{len}_{=k} \wedge \diamond_r (\text{len}_{=0} \wedge p^{\leq} \wedge p^{\geq})), \text{ if } k = m$
$\tau[x, y](s^k(x) = s^m(y)) = \perp, \text{ if } k < m$	$\tau[x, y](P(s^k(x), s^m(x))) =$
$\tau[x, y](s^k(x) = s^m(y)) = \text{len}_{=k-m}, \text{ if } k \geq m$	$\diamond_l \diamond_r (\text{len}_{=m} \wedge \diamond_r (\text{len}_{=k-m} \wedge p^{\geq})), \text{ if } k > m$
$\tau[x, y](s^k(x) < s^m(y)) = \top, \text{ if } k < m$	$\tau[x, y](P(s^k(y), s^m(y))) =$
$\tau[x, y](s^k(x) < s^m(y)) = \text{len}_{>k-m}, \text{ if } k \geq m$	$\diamond_r (\text{len}_{=k} \wedge \diamond_r (\text{len}_{=m-k} \wedge p^{\leq})), \text{ if } k < m$
$\tau[x, y](s^m(y) < s^k(x)) = \perp, \text{ if } k < m$	$\tau[x, y](P(s^k(y), s^m(y))) =$
$\tau[x, y](s^m(y) < s^k(x)) = \text{len}_{<k-m}, \text{ if } k \geq m$	$\diamond_r (\text{len}_{=k} \wedge \diamond_r (\text{len}_{=0} \wedge p^{\leq} \wedge p^{\geq})), \text{ if } k = m$
$\tau[x, y](\neg \alpha) = \neg \tau[x, y](\alpha)$	$\tau[x, y](P(s^k(y), s^m(y))) =$
$\tau[x, y](\alpha \vee \beta) = \tau[x, y](\alpha) \vee \tau[x, y](\beta)$	$\diamond_r (\text{len}_{=m} \wedge \diamond_r (\text{len}_{=k-m} \wedge p^{\geq})), \text{ if } k > m$
$\tau[x, y](\exists x \beta) = \diamond_r (\tau[x, y](\beta)) \vee \square_r \diamond_l (\tau[x, y](\beta))$	$\tau[x, y](P(x, y)) = p^{\leq}$
$\tau[x, y](\exists y \beta) = \diamond_l (\tau[x, y](\beta)) \vee \square_l \diamond_r (\tau[x, y](\beta))$	$\tau[x, y](P(y, x)) = p^{\geq}$

Given an interval model  $M = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), V_M \rangle$ , the corresponding relational model  $\eta(M)$  is a pair  $\langle \mathbb{D}, V_{\eta(M)} \rangle$ , where for all  $p \in \mathcal{AP}$ ,  $V_{\eta(M)}(P) = \{(i, j) \in D \times D : [i, j] \in V_M(p)\}$ . To define the mapping from relational models to interval ones, we associate two propositional letters  $p^{\leq}$  and  $p^{\geq}$  of the interval logic with every binary relation  $P$ . Thus, Given a relational model  $\mathbf{M} = \langle \mathbb{D}, V_{\mathbf{M}} \rangle$ , the corresponding interval model  $\zeta(\mathbf{M})$  is a structure  $\langle \mathbb{D}, \mathbb{I}(\mathbb{D}), V_{\zeta(\mathbf{M})} \rangle$  such that for any binary relation  $P$  and any interval  $[i, j]$ , we have that  $[i, j] \in V_{\zeta(\mathbf{M})}(p^{\leq})$  iff  $(i, j) \in V_{\mathbf{M}}(P)$  and  $[i, j] \in V_{\zeta(\mathbf{M})}(p^{\geq})$  iff  $(j, i) \in V_{\mathbf{M}}(P)$ . We compare the expressive power of an interval modal logic and a first order logic by means of effective translation between formulae and models.

We consider the extension of  $\text{FO}^2[\mathbb{N}, =, <]$  with the successor function  $s$ , denoted by  $\text{FO}^2[\mathbb{N}, =, <, s]$ . The terms of the language  $\text{FO}^2[\mathbb{N}, =, <, s]$  are of the type  $s^k(z)$ , where  $z \in \{x, y\}$  and  $s^k(z)$  denotes  $z$  when  $k = 0$  and  $\underbrace{s(s(\dots s(z)\dots))}_k$  when  $k > 0$ . Moreover,

consider the fragment  $\text{FO}_r^2[\mathbb{N}, =, <, s]$  of  $\text{FO}^2[\mathbb{N}, =, <, s]$  on which the following restriction is imposed: if both variables  $x$  and  $y$  occur in the scope of an occurrence of a binary relation, other than  $=$  and  $<$ , then the successor function  $s$  may not occur in the scope of that occurrence. Thus, e.g., each of the formulae  $s^k(x) = s^m(y)$ ,  $s^k(x) < s^m(y)$ ,  $P(s^k(x), s^m(x))$ ,  $P(x, y)$  belongs to  $\text{FO}_r^2[\mathbb{N}, =, <, s]$ , but none of  $P(x, s(y))$  and  $P(s(x), y)$  belongs there. By using 2-pebble games and a standard model-theoretic argument, one can show that:

$$\text{FO}^2[\mathbb{N}, =, <] \prec \text{FO}_r^2[\mathbb{N}, =, <, s] \prec \text{FO}^2[\mathbb{N}, =, <, s].$$

Here we will extend the result presented in [8] to the language of MPNL. In particular, we show (i) that  $\text{MPNL} \equiv \text{FO}_r^2[\mathbb{N}, =, <, s]$ , (ii) that there is a natural extension of MPNL, denoted here by  $\text{MPNL}^+$  which is functionally complete for  $\text{FO}^2[\mathbb{N}, =, <, s]$ , and (iii) that, perhaps unexpectedly,  $\text{FO}^2[\mathbb{N}, =, <, s]$  (and, therefore,  $\text{MPNL}^+$ ) are already undecidable, which means that the decidability result from [21] cannot be extended by adding one successor function.

First of all consider the following standard translation  $ST_{x,y}$  of MPNL-formulae into  $\text{FO}_r^2[\mathbb{N}, =, <, s]$ , as follows:

$$ST_{x,y}(\varphi) = x \leq y \wedge ST'_{x,y}(\varphi),$$

where  $x, y$  are the two first-order variables in  $\text{FO}_r^2[\mathbb{N}, =, <, s]$ , and:

$$\begin{aligned} ST'_{x,y}(p) &= P(x, y) \\ ST'_{x,y}(\text{len}_{=k}) &= s^k(x) = y \\ ST'_{x,y}(\varphi \vee \psi) &= ST'_{x,y}(\varphi) \vee ST'_{x,y}(\psi) \\ ST'_{x,y}(\neg \varphi) &= \neg ST'_{x,y}(\varphi) \\ ST'_{x,y}(\diamond_l \varphi) &= \exists y (y \leq x \wedge ST'_{y,x}(\varphi)) \\ ST'_{x,y}(\diamond_r \varphi) &= \exists x (x \leq y \wedge ST'_{y,x}(\varphi)). \end{aligned}$$

The fact that a formula  $\varphi$  of MPNL is satisfied on an interval model  $M$  at an interval  $[i, j]$  if and only if  $ST_{x,y}(\varphi)$  is satisfied by substituting  $x$  with  $i$  and  $y$  with  $j$  on the model  $\eta(M)$  can be proved by structural induction on  $\varphi$ . The inverse translation  $\tau$  from  $\text{FO}^2[\mathbb{N}, =, <, s]$  to MPNL is given in Table 1, and we have the following lemma.

**Lemma 14** For every  $\text{FO}_r^2[\mathbb{N}, =, <, s]$ -formula  $\alpha(x, y)$ , every  $\text{FO}_r^2[\mathbb{N}, =, <, s]$ -model  $\mathbf{M} = \langle \mathbb{N}, V_{\mathbf{M}} \rangle$  and every pair  $i, j \in \mathbb{N}$ , with  $i \leq j$ : (i)  $\mathbf{M} \models \alpha(i, j)$  if and only if  $\zeta(\mathbf{M}), [i, j] \Vdash \tau[x, y](\alpha)$ , and (ii)  $\mathbf{M} \models \alpha(j, i)$  if and only if  $\zeta(\mathbf{M}), [i, j] \Vdash \tau[y, x](\alpha)$ .

As a consequence, for every  $\text{FO}_r^2[\mathbb{N}, =, <, s]$ -formula  $\alpha(x, y)$  and every  $\text{FO}_r^2[\mathbb{N}, =, <, s]$ -model  $M = \langle \mathbb{N}, V_M \rangle$ ,  $M \models \forall x \forall y \alpha(x, y)$  if and only if  $\zeta(M) \Vdash \tau[x, y](\alpha) \wedge \tau[y, x](\alpha)$ , which implies the following theorem.

**Theorem 15**  $\text{FO}_r^2[\mathbb{N}, =, <, s] \equiv \text{MPNL}$ .

A natural way to extend MPNL to cover the entire  $\text{FO}^2[\mathbb{N}, =, <, s]$  would be to add diamond modalities that shift respectively the beginning, or the end, of the current interval to the right by a prescribed distance, viz:

- $M, [i, j] \Vdash \diamond_e^{+k} \psi$  iff  $M, [i, j+k] \Vdash \psi$ ;
- $M, [i, j] \Vdash \diamond_b^{+k} \psi$  iff  $(i+k \leq j \text{ and } M, [i+k, j] \Vdash \psi)$  or  $(i+k > j \text{ and } M, [j, i+k] \Vdash \psi)$ ;

We denote the resulting language as  $\text{MPNL}^+$ . It is not difficult to see that the standard translation  $ST'_{x,y}$  of MPNL-formulae into  $\text{FO}^2[\mathbb{N}, =, <, s]$  can be extended to  $\text{MPNL}^+$ , as well as the inverse result, by adding suitable translation clauses to the ones of Table 1.

**Theorem 16**  $\text{FO}^2[\mathbb{N}, =, <, s] \equiv \text{MPNL}^+$ .

Finally, we sketch the proof of the following theorem.

**Theorem 17** The satisfiability problem for  $\text{FO}^2[\mathbb{N}, =, <, s]$ , and consequently for  $\text{MPNL}^+$ , is undecidable.

**Proof.** [sketch] We use a reduction from the *tiling problem for the second octant of the integer plane*, that is, the problem of establishing whether a given finite set of tile types  $\mathcal{T} = \{t_1, \dots, t_k\}$  can tile  $\mathcal{O} = \{(i, j) : i, j \in \mathbb{N} \wedge 0 \leq i \leq j\}$ : using König's lemma, one can prove that a tiling system tiles  $\mathcal{O}$  if and only if it tiles arbitrarily large squares if and only if it tiles  $\mathbb{N} \times \mathbb{N}$  if and only if it tiles  $\mathbb{Z} \times \mathbb{Z}$ . The undecidability of the first one immediately follows from that of the last one [6]. The reduction consists of three main steps: (i) the encoding of an infinite chain that will be used to represent the tiles, (ii) the encoding of the above-neighbor relation by means of a relation denoted by  $Corr$ , and (iii) the encoding of the right-neighbor relation, which will make use of the successor function. Pairs of successive points are used as cells to arrange the tiling: each pair of point of the type  $i, i + 1$  is used either to represent a part of the plane or to separate two consecutive rows of the octant, each one represented by a relation denoted  $Id$ . In the former case, the pair is labeled with the relation  $Tile$ , in the latter case, it is labeled with the relation  $*$ . The encoding is given by the following formulae:

$$\forall x, y \bigwedge_{P \in \mathcal{AP}} (P(x, y) \leftrightarrow P(y, x)) \quad (10)$$

$$\forall x, y (y = s(x) \leftrightarrow *(x, y) \vee Tile(x, y)) \quad (11)$$

$$\forall x, y (*(x, y) \rightarrow \neg Tile(x, y)) \quad (12)$$

$$y = s(x) \wedge *(x, y) \wedge \forall x \exists y (y = s(x)) \quad (13)$$

$$\exists x (x = s(y) \wedge Tile(y, x) \wedge *(s(y), s(x))) \quad (14)$$

$$\exists y (y = s^2(x) \wedge Id(x, y)) \quad (15)$$

$$\forall x, y (Id(x, y) \rightarrow *(y, s(y))) \quad (16)$$

$$\forall x, y (Id(x, y) \rightarrow *(y, s(x))) \quad (17)$$

$$\forall x, y (*(x, y) \rightarrow \exists y (s(x) < y \wedge Id(x, y))) \quad (18)$$

$$\forall x, y (Id(x, y) \rightarrow \overline{Id}_e(s(x), y)) \quad (19)$$

$$\forall x, y (\overline{Id}_e(x, y) \wedge s(x) < y \rightarrow \overline{Id}_e(s(x), y)) \quad (20)$$

$$\forall x, y (Id(x, s(y)) \rightarrow \overline{Id}_b(x, y)) \quad (21)$$

$$\forall x, y (\overline{Id}_b(x, s(y)) \wedge x < y \rightarrow \overline{Id}_b(x, y)) \quad (22)$$

$$\forall x, y ((\overline{Id}_e(x, s(y)) \vee \overline{Id}_d(x, s(y))) \wedge x < y \rightarrow \overline{Id}_d(x, y)) \quad (23)$$

$$\forall x, y ((\overline{Id}_b(x, y) \vee \overline{Id}_e(x, y) \vee \overline{Id}_d(x, y)) \rightarrow \neg Id(x, y)) \quad (24)$$

$$\forall x, y \bigwedge_{\nu, \mu \in \{b, d, e\}, \nu \neq \mu} (\overline{Id}_\nu(x, y) \rightarrow \neg \overline{Id}_\mu(x, y)) \quad (25)$$

$$\forall x, y (Id(x, y) \rightarrow Corr(s(x), s(y))) \quad (26)$$

$$\forall x, y (Corr(x, y) \rightarrow Tile(x, s(x)) \wedge Tile(y, s(y))) \quad (27)$$

$$\forall x, y (Corr(x, y) \wedge *(s(x), s^2(x)) \rightarrow \quad (28)$$

$$Tile(y, s(y)) \wedge Tile(s(y), s^2(y)) \wedge *(s^2(x), s^3(x)))$$

$$\forall x, y (Corr(x, y) \wedge \neg *(s(x), s^2(x)) \rightarrow Corr(s(x), s(y))) \quad (29)$$

$$\forall x, y (Id(x, y) \rightarrow \neg Corr(x, y)) \quad (30)$$

$$\forall x, y (Tile(x, y) \rightarrow \quad (31)$$

$$\bigvee_{T \in \mathcal{T}} T(x, y) \wedge \bigwedge_{T, T' \in \mathcal{T}, T \neq T'} \neg (T(x, y) \wedge T'(x, y)))$$

$$\forall x, y (T(x, y) \wedge Tile(s(x), s(y)) \rightarrow \quad (32)$$

$$\bigvee_{T' \in \mathcal{T}, right(T)=left(T')} T'(s(x), s(y)))$$

$$\forall x, y (Corr(x, y) \wedge T(x, s(x)) \rightarrow \quad (33)$$

$$\bigvee_{T' \in \mathcal{T}, up(T)=down(T')} T'(y, s(y))).$$

Given any set of tiles  $\mathcal{T}$ , the conjunction of the above formulae is satisfiable if and only if  $\mathcal{T}$  can tile  $\mathcal{O}$ . The undecidability of the satisfiability problem for  $FO^2[\mathbb{N}, =, <, s]$  immediately follows. ■

## 6 Concluding remarks

The main results of the paper are the decidability of MPNL, its expressive equivalence to the fragment  $FO_r^2[\mathbb{N}, =, <, s]$  of  $FO^2[\mathbb{N}, =, <, s]$ , and the undecidability of  $FO^2[\mathbb{N}, =, <, s]$ . These results together position MPNL very close to the decidability/undecidability border and it would be interesting to know whether it can be further extended, syntactically or semantically, in a natural way, still preserving decidability. In particular, the decidability of both MPNL interpreted over the integers and the extension of MPNL with rational constraints for interval lengths, interpreted over the rational numbers, is natural to expect. Efficient model-checking for MPNL on natural numbers is another technical challenge ahead.

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